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On the sum of a Special ${}_4F_3$

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1. Recently Carlitz [1, p. 90] has obtained the sum of a particular hypergeometric series ${}_4F_3\ (1)$ in the form

(1) $${}_4F_3\left[\begin{matrix}-n, \frac{1}{2}+\frac{1}{2}\alpha, 1+\frac{1}{2}\alpha, \lambda+n \\ 1+\alpha, \frac{1}{2}+\frac{1}{2}\lambda, 1+\frac{1}{2}\lambda\end{matrix}\right] = \frac{\lambda\,(\lambda-\alpha)_n}{(\lambda+2n)\,(\lambda)_n}$$

We consider the sum

$$A_n(\alpha,\lambda) = \sum_{r=0}^{n} \frac{(-n)_r\,(\alpha)_{2r}\,(\lambda+n)_r}{r!\,(1+\alpha)_r\,(\lambda)_{2r}}$$

$$= {}_4F_3\left[\begin{matrix}-n, \frac{1}{2}\alpha, \frac{1}{2}+\frac{1}{2}\alpha, \lambda+n \\ 1+\alpha, \frac{1}{2}\lambda, \frac{1}{2}+\frac{1}{2}\lambda\end{matrix}\right],$$

where the ${}_4F_3(1)$ series is a Saalschützian* and n is a non-negative integer. It will be proved that

(2) $$A_n(\alpha, \lambda) = \frac{(\lambda-\alpha)_n}{(\lambda)_n}$$

Put

$$f_n(\alpha,\lambda) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} (\alpha+r+1)_{r-1}\,(\lambda+2r)_{n-r}\,\alpha$$

where it is understood that the term on the right corresponding to $r = 0$ is $(\lambda)_n$.

Then it is easily verified that (2) is equivalent to

(3) $$f_n(\alpha, \lambda) = (\lambda-\alpha)_n$$

Now we have

$$f_n(\alpha+1, \lambda) - f_n(\alpha, \lambda)$$

*Notice that the series we discuss here is slightly different from that in (1).

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} (\lambda + 2r)_{n-r} [(1+\alpha)(\alpha + 2 + r)_{r-1} - \alpha(\alpha + r + 1)_{r-1}]$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} (\lambda+2r)_{n-r} (2 + \alpha + r)_{r-2} [(1+\alpha)(\alpha + 2r) - \alpha(\alpha - r + 1)]$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} (\lambda + 2r)_{n-r} (2 + \alpha + r)_{r-2} [r(2+\alpha)]$$

$$= (-n) \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} (\lambda + 2 + 2r)_{n-r-1} (1 + \alpha + r)_{r-3} (\alpha + 2)$$

$$= -n f_{n-1} (\alpha + 2, \lambda + 2)$$

which implies

(4)
$$f_n(\alpha + 1, \lambda) - f_n(\alpha, \lambda) = -n f_{n-1} (\alpha + 2, \lambda + 2)$$

Clearly (3) holds when $n = 0$. Assuming the truth of (3) for all values less then n, it follows from (4) that

(5)
$$f_n(\alpha + 1, \lambda) - f_n(\alpha, \lambda) = -n (\lambda - \alpha)_{n-1}$$

If we keep λ fixed and note that $f_n(\alpha, \lambda)$ is a polynomial in α, it is clear from (5) that

(6)
$$f_n(\alpha, \lambda) = (\lambda - \alpha)_n + g_n(\lambda)$$

where $g_n(\lambda)$ is independent of α. We now take $\alpha = \lambda$, so that (6) becomes

$$f_n(\lambda, \lambda) = g_n(\lambda) \qquad (\lambda > 0).$$

Now

$$f_n(\lambda, \lambda) = (\lambda)_n A_n(\lambda, \lambda)$$

$$= (\lambda)_n \sum_{r=0}^{n} \frac{(-n)_r (\lambda + n)_r}{r! (1 + \lambda)_r}$$

Since

$$\sum_{r=0}^{n} \frac{(-n)^r (\lambda + n)_r}{r! (1 + \lambda)_r} = \frac{(1-n)_n}{(1+\lambda)_n} = 0$$

for $n > 0$, therefore $g_n(\lambda) = 0$ and (6) reduces to (3)

2. We have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} A_n(\alpha, \lambda) t^n$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} t^n \sum_{r=0}^{n} \frac{(\alpha)_{2r} (\lambda + n)_r}{r! (1 + \alpha)_r (\lambda)_{2r}} (-1)_r \binom{n}{r}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(\lambda)_{n+r} (\alpha)_{2r} (-1)^r}{r!\,(n-r)!\,(1+\alpha)_r (\lambda)_{2r}} t^n$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda)_{n+2r} (\alpha)_{2r} (-1)^r}{r!\,n!\,(1+\alpha)_r (\lambda)_{2r}} t^{n+r}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_{2r}}{r!\,(1+\alpha)_r} (-t)^r (1-t)^{-\lambda-2r}$$

$$= (1-t)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\alpha)_{2r}}{r!\,(1+\alpha)_r} \left[\frac{-t}{(1-t)^2}\right]^r$$

which must be equal to

$$\sum_{n=0}^{\infty} \frac{(\lambda-\alpha)_n}{n!} t^n$$

$$= (1-t)^{\alpha-\lambda}$$

Thus we get

$$\sum_{r=0}^{\infty} \frac{(\alpha)_{2r}}{r!\,(1+\alpha)_r} \left[\frac{-t}{(1-t)^2}\right]^r = (1-t)^{\alpha}$$

If we put $\dfrac{-4t}{(1-t)^2} = z$, then

$$1 - t = [\tfrac{1}{2} + \tfrac{1}{2}(1-z)^{\frac{1}{2}}]^{-1}$$

and we obtain the result [2 ; p. 101(6)].

3. The result (1) can also be obtained by using (2) in the known transformation [3 ; p. 56]

$${}_4F_3\left[\begin{matrix} a, b, c, -m \\ \alpha, \beta, \gamma \end{matrix}\right] = \frac{(\beta-a)_m (\alpha+\beta-b-c)_m}{(\beta)_m (\alpha+\beta-a-b-c)_m} {}_4F_3\left[\begin{matrix} a, \alpha-b, \alpha-c, -m ; \\ \alpha, a+\beta-b-c, \alpha+\gamma-b- \end{matrix}\right.$$

which holds when $\alpha + \beta + \gamma = 1 + a + b + c - m$.

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Algebraie formulations of a topological space

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Introduction

Pervin[1] has derived a theory of a topological space from a quasi proximity space (X, δ) where δ is a binary relation on the power set $p(X)$ of a set X satisfying the properties listed in section 3. It is then natural to pose the problem :

Is it not possible to replace the relation δ by a binary operation on $p(X)$ so that the resulting algebraic structure may be equivalent to a quasi proximity structure and therefore, to a topological structure on X ?

The paper proposes to give a positive solution of the problem by setting up two algebraic formulations of a topological space. Such an algebraic structure defining a topological space may here be called an algebraic space which will :

(1) Contain Krishna Murti's commutative topological algebra[2],

(2) be a generalisation of the usual topological space[3].

1. *An algebraic space.* An algebraic space $(X, 9)$ of order α is a set X together with a binary operation 9 on $p(X)$ satisfying the properties.

I 1 $A \, 9 \, A = A$ for each A

I 2 $A \, 9 \, 1 = 1$ for each A

I 3 There exists an initial ordinal number α such that 9 is distributed over the product of each λ sequence in $p(X)$ whenever λ is less than α *i e.*

$$A \, 9 \left(\Pi_{\zeta \epsilon \lambda \epsilon \alpha} \beta_\zeta \right) = \Pi_{\zeta \epsilon \lambda \epsilon \alpha} \; A \, 9 \, \beta_\zeta.$$

I 4 $A \, 9 \, (O \, 9 \, B) = A \, 9 \, B$ for each A and B.

Definition 1·1 $(A' \, 9 \, B')' = A \, s \, B$.

Theorem 1·1. In an algebraic space $(X, 9)$ the operation s defined above satisfies the properties :

C 1 $A \, s \, A = A$

C 2 $A \, s \, O = O$

C 3 $A \, s \sum_{\zeta \epsilon \lambda \epsilon \alpha} B_\zeta = \sum_{\zeta \epsilon \lambda \epsilon \alpha} A \, s \, B_\zeta$

C 4 $A \, s \, (1 \, s \, B) = A \, s \, B$

Proof : $A \, s \, A = (A' \, 9 \, A')' = A'' = A$ (C1)

$A \, s \, O = (A' \, 91)' = 1' = O$ (C2)

$$A \mathbin{s} \Sigma B_{\zeta} = (A' \mathbin{9} \Pi B'_{\zeta})' = (\Pi A' \mathbin{9} B'_{\zeta})' = \Sigma A \mathbin{s} B_{\zeta}$$

$$A \mathbin{s} (1 \mathbin{s} B) = (A' \mathbin{9} (O \mathbin{9} B'))' = (A' \mathbin{9} B')' = A \mathbin{s} B$$

Definition 1·2. A set X together with a binary operation s satisfying the propertie C1 – C4 listed above is called a dual space of an algebraic space.

Theorem 1·2. A dual space (X, s) is an algebraic space provided that

$$A \mathbin{s} B = (A' \mathbin{9} B')'$$

As the operations 9 and s are dual, the proof at once follows from theorer 1·1 by the principle of duality.

2. *Topological structure in an algebraic space.* In an algebraic space $(X, 9)$, th operation 9 will induce a topology in X by setting : $A \mathbin{9} B = A + B^{i}$, where B is interior B.

clearly $O \mathbin{9} B = B^{i}$.

Theorem 2·1. An algebraic space $(X, 9)$ of order α is a topological space c order α satisfying generalised interior axioms, and conversely provided tha $A \mathbin{9} B = A + B^{i}$.

Proof: Let $(X, 9)$ be an algebraic space of order α. Then

$$A = A \mathbin{9} A = A + A^{i} \Rightarrow A^{i} \subset A$$

$$1 = O \mathbin{9} 1 = 1^{i}$$

$$\left(\Pi_{\zeta \varepsilon \lambda} B_{\zeta} \right)^{i} = O \mathbin{9} (\Pi B_{\zeta}) = \Pi (O \mathbin{9} B_{\zeta}) = \Pi B^{i}_{\zeta}$$

A

Conversely let $(X, ^{(i)})$ be a topological space of order α. Then

$$A \mathbin{9} A = A + A^{i} = A \qquad (91)$$

$$A \mathbin{9} 1 = A + 1^{i} = A + 1 = 1 \qquad (92)$$

$$A \mathbin{9} (\Pi B_{\zeta}) = A + (\Pi B_{\zeta})^{i} = A + \Pi (B_{\zeta})^{i} = \Pi (A + B_{\zeta})^{i}$$
$$= \Pi A \mathbin{9} B_{\zeta} \qquad (93)$$

$$A \mathbin{9} (O \mathbin{9} B) = A \mathbin{9} (B^{i}) = A + B^{ii} = A + B^{i} = A \mathbin{9} B \qquad (94)$$

Theorem 2·2. A dual space (X, s) of order α is a topological space of order satifying generalised Kuratowski's closure axioms and conversely provided tha $A \mathbin{s} B = A . B^{c}$.

The proof at once follows from theorem 2·1 from the principle of dualit as s is dual to 9 and$^{(c)}$ is dual to $^{(i)}$.

Theorem 2·3. A dual space (X, s) of order ω is Krishna Murti's topologica algebra ; and conversely provided that $A \mathbin{s} B = A B^{c} + B A^{c}$.

The proof is trivial.

3. *Proximity structure in* $(X, 9)$. A quasi proximity structure can be introduced in an algebraic space $(X, 9)$ by setting :

$$(A, B) \ \varepsilon \ \delta \ \textit{iff} \ A' \mathbin{9} B' \neq 1.$$

Theorem 3·1. An algebraic space $(X, 9)$ of order α is a quasi proximity space of order α satisfying generalised quasi proximity axioms of Pervin ; and conversely, provided that $(A\ B) \varepsilon \delta$ *iff* $A'\ 9\ B' \neq 1$.

Q 1 $(A, O) \bar{\varepsilon} \delta$.

For $A'\ 9\ 1 = A' + 1^i = A' + 1 = 1$

Q 2 $(x, x) \varepsilon \delta$

For $x'\ 9\ x' = x' \neq 1$.

Q 3 $\left(A, \Sigma_{\zeta \varepsilon \lambda} B\zeta\right) \bar{\varepsilon} \delta$ *iff* $(A, B\zeta) \bar{\varepsilon} \delta$ for each $\zeta \varepsilon \lambda$

For $(A, \Sigma B_{\zeta}) \bar{\varepsilon} \delta \Rightarrow A'\ 9\ \Pi B'_{\zeta} = 1 \Rightarrow A'\ 9\ B'_{\zeta} = 1$ for each ζ

$\Rightarrow (A, B_{\zeta}) \bar{\varepsilon} \delta$ for each ζ.

Q 4 If $(A, B) \bar{\varepsilon} \delta$, then there exists a pair of disjoint sets U and V such that $(A, U') \bar{\varepsilon} \delta$ and $(V', B) \bar{\varepsilon} \delta$.

For $(A, B) \bar{\varepsilon} \Rightarrow A' + (B')^i = 1$.

Let $U = (B')^i$ and $V = ((B')^i)'$

Then $A'\ 9\ U = A' + (B')^{ii} = A' + (B')^i = 1 \Rightarrow (A, U) \bar{\varepsilon} \delta$.

Also $((B')^i)'\ 9\ B' = ((B')^i)' + (B')^i = 1 \Rightarrow (V', B) \varepsilon \delta$

For the converse, let (X, δ) be a proximity space of order α.

Then $(A)^i = \{x \mid x \bar{\delta} A'\}$, where $\bar{\delta}$ is negated δ.

It can easily be verified that (X, δ) satisfies the generalised interior anxioms.

Theorem 3·2. A dual space (X, s) of order α is a quasi proximity space of the same order and conversely provided that $(A, B) \varepsilon \delta$ *iff* $A\ s\ B \neq 0$.

The proof of the theorem at once follows from 3·1 from the principle of duality.

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Some integrals involving generalized Legendre associated functions and H-function

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1. Fox [4, p. 408], introduced the H-function in the form of Mellin-Barnes typ integral as

$$(1\cdot1) \qquad \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\ \prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\ \prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\, x^s\, ds,$$

where x is not equal to zero and empty product is interpreted as unity ; p, q, and n are integers satisfying $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$;

α_j $(j = 1, \ldots, p)$, β_j $(j = 1, \ldots, q)$ are positive numbers and a_j $(j = 1, \ldots, p)$ b_j $(j = 1, \ldots, q)$ are complex numbers such that no pole of $\Gamma(b_h - \beta_h s)$ $(h = 1, \ldots, m)$ coincides with any pole of $\Gamma(1 - a_i + \alpha_i s)$ $(i = 1, 2, \ldots, n)$ i.e

$$(1\cdot2) \qquad \alpha_i\,(b_h+\nu) \neq (a_i-\eta-1)\,\beta_h$$

$$(\nu, \eta = 0, 1, \ldots ; h = 1, \ldots, m ; i = 1, \ldots, n)$$

Further the contour T runs from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points

$$(1\cdot3) \qquad s = \frac{b_h+\nu}{\beta_h} \ (h = 1, \ldots, m ; \nu = 0, 1, \ldots),$$

which are poles of $\Gamma(b_h - \beta_h s)$ $(h = 1, \ldots, m)$ lie to the right and the points

$$(1\cdot4) \qquad s = \frac{(a_i-\eta-1)}{\alpha_i} \ (i = 1, \ldots, n ; \eta = 0, 1, \ldots),$$

which are the poles of $\Gamma(1 - a_i + \alpha_i s)$ $(i = 1, \ldots, n)$ lie to the left of T. Su a contour is possible on account of (1·2). These assumptions for the H-functi will be adhered to throughout this paper.

Recently Gupta and Jain[5] have denoted (1·1) symbolically as

$$(1\cdot5) \qquad H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$

and in a more compact form by

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right], \tag{1·6}$$

where $\{(f_r, \gamma_r)\}$ stands for the set of the parameters $(f_1, \gamma_1), \ldots\ldots\ldots,$ (f_r, γ_r).

According to Braaksma [2, p. 278]

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] = O(|x|^{\alpha}) \text{ for small } x,$$

where $\sum_1^p (\alpha_j) - \sum_1^q (\beta_j) \leqslant 0$, and $\alpha = Re\left(\frac{b_h}{\beta_h}\right)$ $(h = 1, \ldots, m)$ and

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] = O(|x|^{\beta}) \text{ for large } x,$$

where $\sum_1^p (\alpha_j) - \sum_1^q (\beta_j) < 0$, $\sum_1^n (\alpha_j) - \sum_{n+1}^p (\alpha_j) + \sum_1^m (\beta_j) - \sum_{m+1}^q (\beta_j) = \lambda > 0$,

$$|\arg x| < \tfrac{1}{2}\lambda\pi \text{ and } \beta = Re\left(\frac{a_i - 1}{\alpha_i}\right) (i = 1, \ldots, n).$$

In[6], Meulenbeld and Kuipers defined generalized Legendre associated function $P_k^{m,n}(z)$ for general values (real or complex) of the parameters in terms of (Pochhammer's) integrals, and these have been transformed into hypergeometric functions. Assumptions on the parameters k, m, n and z gave rise to further transformations. For example if k is an integer $\geqslant 0$,

$k - \frac{m-n}{2}$ is a non-negative integer, then for $|1 - x| < 2$, we have

$$P_{k-\frac{m-n}{2}}^{m,n}(x) = \frac{1}{\Gamma(1-m)}(1+x)^{\frac{n}{2}}(1-x)^{-\frac{m}{2}} F\left[\begin{matrix}-k, k-m+n+1; \frac{1-x}{2}\\ 1-m\end{matrix}\right].$$

The object of this paper is to evaluate some integrals involving product of the generalised Legendre associated functions and the H-function. As the H-function is a very general function, we get on specializing the parameters of the integrals established in section 3, many integrals, some of which are known and others are believed to be new.

2. In this section, we state the results given by Meulenbeld and Robin[8], Gupta and Jain[5]; which will be used in our present work.

Definite integrals (36), (37)*, (38)† and (52) of [8, p. 343 and p. 345],

$$\int_{-1}^{1}(1-x)^{-\frac{m}{2}}(1+x)^{\sigma} P_{k-\frac{m-n}{2}}^{m,n}(x)\, dx \tag{2·1}$$

$$= \frac{2^{\sigma-m+\frac{n}{2}+1}\,\Gamma(\sigma+\frac{n}{2}+1)\,\Gamma(\sigma-\frac{n}{2}+1)}{\Gamma(\sigma-\frac{n}{2}-k+1)\,\Gamma(\sigma-m+\frac{n}{2}+k+2)},$$

provided $Re(m) < 1$, $Re\,(\sigma+\frac{n}{2}) > -1$.

(2·2)
$$\int_{-1}^{1} (1-x)^{\rho}\,(1+x)^{\frac{n}{2}}\,P^{m,\,n}_{k-\frac{m-n}{2}}(x)\,dx$$

$$= \frac{2^{\rho-\frac{m}{2}+n+1}\,\Gamma(n+k+1)\,\Gamma(\rho-\frac{m}{2}+1)\,\Gamma(-\rho-\frac{m}{2}+k)}{\Gamma(k-m+1)\,\Gamma(-\rho-\frac{m}{2})\,\Gamma(\rho-\frac{m}{2}+n+k+2)},$$

where $Re\,(\rho-\frac{m}{2}) > -1$, $Re\,(n) > -1$.

(2·3)
$$\int_{-1}^{1} (1-x)^{\rho}\,(1+x)^{\sigma}\,P^{m,\,n}_{k-\frac{m-n}{2}}(x)\,dx$$

$$= \frac{2^{\rho+\sigma-\frac{m-n}{2}+1}\,\Gamma(\rho-\frac{m}{2}+1)\,\Gamma(\sigma+\frac{n}{2}+1)}{\Gamma(1-m)\,\Gamma(\rho+\sigma-\frac{m-n}{2}+2)} \times$$

$$\times\ {}_3F_2\left(-k,\ n-m+k+1,\ \rho-\frac{m}{2}+1\ ;\ 1-m,\ \rho+\sigma-\frac{m-n}{2}+2\ ;\ 1\right),$$

provided $Re\left(\rho-\frac{m}{2}\right) > -1$, $Re\left(\sigma+\frac{n}{2}\right) > -1$.

(2·4)
$$\int_{-1}^{1} (1-x)^{\tau+\frac{m+\rho}{2}}\,(1+x)^{\frac{n-\sigma}{2}}\,P^{m,\,n}_{k-\frac{m-n}{2}}(x)\,P^{\rho,\,\sigma}_{l-\frac{\rho-\sigma}{2}}(x)\,dx$$

$$= \frac{2^{\tau+n+1}\,\Gamma(k-m-\tau)\,\Gamma(n+k+1)\,\Gamma(\tau+1)}{\Gamma(k-m+1)\,\Gamma(-m-\tau)\,\Gamma(1-\rho)\,\Gamma(n+\tau+k+2)} \times$$

$$\times\ {}_4F_3\,(-l,\ \sigma-\rho+l+1,\ \tau+1,\ \tau+m+1\ ;\ 1-\rho,\ n+\tau+k+2,\ \tau+m-k+1;\ 1),$$

where $Re(n) > -1$, $Re(\tau) > -1$.

Relation between G and H–functions

(2·5)
$$H^{m,\,n}_{p,\,q}\left[x\,\middle|\,\begin{matrix}\{(a_p,1)\}\\ \{(b_q,1)\}\end{matrix}\right] = G^{m,\,n}_{p,\,q}\left(x\,\middle|\,\begin{matrix}a_1,\ldots\ldots,a_p\\ b_1,\ldots\ldots,b_q\end{matrix}\right).$$

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*$\Gamma(\rho+\frac{m}{2}+n+k+2)$, a factor in the right-hand side denominator of [8, p. 343(37)], should be $\Gamma(\rho-\frac{m}{2}+n+k+2)$, as mentioned in the result (2·2).

†$\rho+\sigma-\frac{m+n}{2}+2$, a parameter in the hypergeometric function of [8, p. 343(38)], should be $\rho+\sigma-\frac{m-n}{2}+2$, as given in the result (2·3).

$$(2{\cdot}6) \qquad H^{1,\,p}_{p,\,q+1}\left[x \,\middle|\, \begin{matrix}\{(1-a_p,\alpha_p)\} \\ (0,1),\{(1-b_q,\beta_q)\}\end{matrix}\right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j s)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j s)} \frac{(-x)^r}{r\,!}.$$

The above series was studied by Wright [10, p. 287] and has been called as Wright's generalised hypergeometric function and is denoted by the symbol

$${}_p\psi_q\left[\begin{matrix}\{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\}\end{matrix};-x\right].$$

$$(2{\cdot}7) \qquad H^{1,\,0}_{0,\,2}\left[x \,\middle|\, (0,1),(-\nu,\mu)\right] = \sum_{r=0}^{\infty} \frac{(-x)^r}{r\,!\,\Gamma(1+\nu+\mu r)} = J_\nu^\mu(x)$$

where $J_\nu^\mu(x)$ is Maitland's generalised Bessel function [11, p. 257].

$$(2{\cdot}8) \qquad H^{1,\,1}_{1,\,1}\left[x \,\middle|\, \begin{matrix}(1-\nu,1) \\ (0,1)\end{matrix}\right] = \Gamma(\nu)(1+x)^{-\nu} = \Gamma(\nu)\,{}_1F_0(\nu;;-x).$$

3. In what follows the notation $\left(a+\begin{vmatrix}r_1\\ \vdots \\ r_n\end{vmatrix},\delta\right)$, will stand for the set of the parameters $(a+r_1,\delta),\ldots\ldots,(a+r_n,\delta)$.

The first integral to be evaluated is

$$(3{\cdot}1) \quad \int_{-1}^{1}(1-x)^{\tau+\frac{m+\rho}{2}}(1+x)^{\frac{n-\sigma}{2}}P^{m,\,n}_{k-\frac{m-n}{2}}(x)\,P^{\rho,\,\sigma}_{l-\frac{\rho-\sigma}{2}}(x)\,H^{p,\,q}_{r,\,s}\left[z(1-x)^\delta \,\middle|\, \begin{matrix}\{(a_r,\alpha_r)\} \\ \{(b_s,\beta_s)\}\end{matrix}\right]dx,$$

$$= \frac{2^{\tau+n+1}\,\Gamma(n+k+1)}{\Gamma(k-m+1)}\sum_{u=0}^{l}\frac{(-l)_u\,(\sigma-\rho+l+1)_u}{u\,!\,\Gamma(1-\rho+u)} \times$$

$$\times H^{p+1,\,q+3}_{r+4,\,s+4}\left[2^\delta z \,\middle|\, \begin{matrix}\left(-u+\begin{vmatrix}-\tau \\ \tau-m\end{vmatrix},\delta\right),(+k-\tau-m,\delta),\{(a_r,\alpha_r)\},(-m-\tau,\delta) \\ (-k-m-\tau,\delta),\{(b_s,\beta_s)\},\left(-\tau+\begin{vmatrix}-m \\ -n-k-u-1 \\ k-m-u\end{vmatrix},\delta\right)\end{matrix}\right]$$

where $\delta>0$; $k-\frac{m-n}{2}$, $l-\frac{\rho-\sigma}{2}$ are positive integers, k and l are integers≥ 0, $\sum_1^q(\alpha_j)-\sum_{q+1}^r(\alpha_j)+\sum_1^p(\beta_j)-\sum_{p+1}^s(\beta_j)\equiv\lambda>0$, $|\arg z|<\frac{1}{2}\lambda\pi$, $\sum_1^s(\beta_j)-\sum_1^r(\alpha_j)\geq 0$

$Re(n)>-1$, $Re(\tau+\delta b_j/\beta_j)>-1$ $(j=1,2,\ldots,p)$.

Proof : To establish the integral (3·1), expressing the H-function in Mellin-Barnes type of integral (1·1) and interchanging the order of integration, which is justifiable due to the absolute convergence of the integrals involved in the process, we get

$$\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{p}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{q}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=p+1}^{s}\Gamma(1-b_j+\beta_j\xi)\prod_{j=q+1}^{r}\Gamma(a_j-\alpha_j\xi)} z^{\xi}\times$$

$$\times\int_{-1}^{1}(1-x)^{\tau+\delta\xi+\frac{m+\rho}{2}}(1+x)^{\frac{n-\sigma}{2}} P_{k-\frac{m-n}{2}}^{m,\,n}(x)\; P_{l-\frac{\rho-\sigma}{2}}^{\rho,\,\sigma}(x)\,dx\,d\xi;$$

evaluating the inner integral with the help of (2·4), it reduces to

$$\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{p}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{q}\Gamma(1-a_j+\alpha_j\xi)\,2^{\tau+\delta\xi+n+1}\,\Gamma(k-m-\tau-\delta\xi)\,\Gamma(n+k+1)}{\prod_{j=p+1}^{s}\Gamma(1-b_j+\beta_j\xi)\prod_{j=q+1}^{r}\Gamma(a_j-\alpha_j\xi)\,\Gamma(k-m+1)\,\Gamma(-m-\tau-\delta\xi)\,\Gamma(1-\rho)}\times$$

$$\times\frac{\Gamma(\tau+1+\delta\xi)\,z^{\xi}}{\Gamma(n+\tau+k+2+\delta\xi)}\sum_{u=0}^{l}\frac{(-l)_u\,(\sigma-\rho+l+1)_u\,\Gamma(\tau+1+u+\delta\xi)}{u!\,(1-\rho)_u\,\Gamma(\tau+\delta\xi+1)\,\Gamma(\tau+m+1+\delta\xi)}\times$$

$$\times\frac{\Gamma(\tau+m+1+u+\delta\xi)\,\Gamma(\tau+n+k+2+\delta\xi)\,\Gamma(\tau+m-k+1+\delta\xi)}{\Gamma(\tau+n+k+u+2+\delta\xi)\,\Gamma(\tau+m-k+u+1+\delta\xi)}d\xi,$$

now changing the order of integration and summation, in view of [3, p. 500]; which is permissible under the conditions given in (2·4) and (3·1), using [9, p. 23(3)] and definition (1·1) of the H-function, the value of the integral is obtained.

Proceeding as in (3·1), the following integrals can be established with the help of (2·1), (2·2) and (2·3) respectively.

$$(3\cdot2)\quad \int_{-1}^{1}(1-x)^{-\frac{m}{2}}(1+x)^{\sigma}P_{k-\frac{m-n}{2}}^{m,\,n}(x)\,H_{r,\,s}^{p,\,q}\left[z\,(1+x)^{\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right]dx$$

$$=2^{\sigma-m+\frac{n}{2}+1}\,H_{r+2,\,s+2}^{p,\,q+2}\left[2^{\delta}z\,\middle|\,\begin{matrix}(-\sigma-\frac{n}{2},\delta),\,(\frac{n}{2}-\sigma,\delta),\,\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},\,(\frac{n}{2}+k-\sigma,\delta),\,(m-\sigma-k-\frac{n}{2}-1,\delta)\end{matrix}\right]$$

provided $\delta>0$; $k-\frac{m-n}{2}$ is a positive integer, k is an integer $\geqslant 0$,

$\sum_{1}^{q}(\alpha_j)-\sum_{q+1}^{r}(\alpha_j)+\sum_{1}^{p}(\beta_j)-\sum_{p+1}^{s}(\beta_j)\equiv\lambda>0$, $|\arg z|<\frac{1}{2}\lambda\pi$, $\sum_{1}^{s}(\beta_j)-\sum_{1}^{r}(\alpha_j)\geqslant 0$,

$Re(m) < 1,\ Re(\sigma + \frac{n}{2} + \delta\, b_j/\beta_j) > -1\ (j = 1, \ldots, p).$

(3·3) $$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\frac{n}{2}}\, P^{m,n}_{k-\frac{m-n}{2}}(x)\, H^{p,q}_{r,s}\left[z(1-x)^{\delta} \middle| \begin{matrix}\{(a_r, \alpha_r)\}\\ \{(b_s, \beta_s)\}\end{matrix}\right] dx$$

$$= \frac{2^{\rho-\frac{m}{2}+n+1}\,\Gamma(n+k+1)}{\Gamma(k-m+1)}\, H^{p+1,q+1}_{r+2,s+2}\left[2^{\delta} z \middle| \begin{matrix}\left(\frac{m}{2}-\rho, \delta\right), \{(a_r,\alpha_r)\}, \left(-\frac{m}{2}-\rho, \delta\right)\\ \left(k-\frac{m}{2}-\rho, \delta\right), \{(b_s,\beta_s)\}, \left(-1-k+\frac{m}{2}-\rho-n, \delta\right)\end{matrix}\right]$$

where $\delta > 0$; $k - \frac{m-n}{2}$ is a positive integer, k is an integer $\geqslant 0$,

$\sum_1^q (a_j) - \sum_{q+1}^r (\alpha_j) + \sum_1^p (\beta_j) - \sum_{p+1}^s (\beta_j) \equiv \lambda > 0,\ |\arg z| < \frac{1}{2}\lambda\pi,\ \sum_1^s (\beta_j) - \sum_1^r (\alpha_j) \geqslant 0,$

$Re(n) > -1,\ Re(\rho - \frac{1}{2} m + \delta b_j/\beta_j) > -1\ (j = 1, \ldots\ldots, p).$

(3·4) $$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma}\, P^{m,n}_{k-\frac{m-n}{2}}(x)\, H^{p,q}_{r,s}\left[z(1-x^2)^{\delta} \middle| \begin{matrix}\{(a_r, \alpha_r)\}\\ \{(b_s, \beta_s)\}\end{matrix}\right] dx$$

$$= 2^{\rho+\sigma-\frac{m-n}{2}+1} \sum_{u=0}^{k} \frac{(-k)_u\,(n-m+k+1)_u}{u\,!\,\Gamma(1-m+u)} \times$$

$$\times H^{p,q+2}_{r+2,s+1}\left[z.2^{2\delta} \middle| \begin{matrix}\left(-\frac{n}{2}-\sigma, \delta\right), \left(\frac{m}{2}-\rho-u, \delta\right), \{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}, \left(\frac{m-n}{2}-\rho-\sigma-1-u, 2\delta\right)\end{matrix}\right]$$

provided $\delta > 0$, $k - \frac{m-n}{2}$ is a positive integer, k is an integer $\geqslant 0$,

$\sum_1^q (a_j) - \sum_{q+1}^r (a_j) + \sum_1^p (\beta_j) - \sum_{p+1}^s (\beta_j) \equiv \lambda > 0,\ |\arg z| < \frac{1}{2}\lambda\pi,\ \sum_1^s (\beta_j) - \sum_1^r (\alpha_j) \geqslant 0,$

$Re(\rho - \frac{1}{2} m + \delta b_j/\beta_j) > -1,\ Re(\sigma + \frac{1}{2} n + \delta b_j/\beta_j) > -1\ (j = 1, 2, \ldots, p).$

4. *Particular Cases* : Various particular cases of the integrals evaluated above in section 3, can be obtained. Some interesting cases are given here.

In (3·1) replacing p, q, r, s respectively by $1, p, p, q+1$ and choosing the other parameters suitably in view of (2·6) we obtain

(4·1) $$\int_{-1}^{1} (1-x)^{\tau+\frac{m+\rho}{2}}(1+x)^{\frac{n-\sigma}{2}}\, P^{m,n}_{k-\frac{m-n}{2}}(x)\, P^{\rho,\sigma}_{l-\frac{\rho-\sigma}{2}}(x)\ {}_p\psi_q\left[\begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix};-z(1-x)^{\delta}\right] dx$$

$$= \frac{2^{\tau+n+1}\,\Gamma(k+n+1)}{\Gamma(k-m+1)} \sum_{u=0}^{l} \frac{(-l)_u\,(\sigma-\rho+l+1)_u}{u\,!\,\Gamma(1-\rho+u)} \times$$

$$\times H^{2,\,p+3}_{p+4,\,q+5}\left[2^{\delta}z \,\middle|\, \begin{matrix} (-u-\tau,\delta),(\tau-m-u,\delta),(k-\tau-m,\delta),\{(1-a_p,\alpha_p)\},(-m-\tau,\delta) \\ (-k-m-\tau,\delta),(0,1),\{(1-b_q,\beta_q)\},\left(-\tau+\begin{vmatrix} -m \\ -n-k-u-1 \\ k-m-u \end{vmatrix},\delta\right) \end{matrix}\right]$$

where $\delta > 0$; $k - \frac{m-n}{2}$, $l - \frac{\rho-\sigma}{2}$ are positive integers, k and l are integers $\geqslant 0$, $\sum_1^p (\alpha_j) + 1 - \sum_1^q (\beta_j) \equiv \lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$, $1 + \sum_1^q (\beta_j) - \sum_1^p (\alpha_j) \geqslant 0$ $Re(n) > -1$, $Re(\tau) > -1$.

In (3·1), setting $p = 1, q = r = 0, s = 2, b_1 = 0, b_2 = -\nu, \beta_1 = 1$ and $\beta_2 = \mu$, by virtue of (2·7), it reduces to

(4·2) $$\int_{-1}^{1} (1-x)^{\tau+\frac{m+\rho}{2}} (1+x)^{\frac{n-\sigma}{2}} P^{m,\,n}_{k-\frac{m-n}{2}}(x)\, P^{\rho,\sigma}_{l-\frac{\rho-\sigma}{2}}(x)\, J_{\nu}^{\mu}\,(z(1-x)^{\delta})\, dx$$

$$= \frac{2^{\tau+n+1}\,\Gamma(k+n+1)}{\Gamma(k-m+1)} \sum_{u=0}^{l} \frac{(-l)_u\,(\sigma-\rho+l+1)_u}{u\,!\,\Gamma(1-\rho+u)} \times$$

$$\times H^{2,\,3}_{4,\,6}\left[2^{\delta}z \,\middle|\, \begin{matrix} (-\tau-u,\delta),(\tau-m-u,\delta),(k-\tau-m,\delta),(-m-\tau,\delta) \\ (-k-m-\tau,\delta),(0,1),(-\nu,\mu),\left(-\tau+\begin{vmatrix} -m \\ -n-k-u-1 \\ k-m-u \end{vmatrix},\delta\right) \end{matrix}\right]$$

where $\delta > 0$; $k - \frac{m-n}{2}$, $l - \frac{\rho-\sigma}{2}$ are positive integers, k and l are integers $\geqslant 0$, $(1-\mu) > 0$, $(1+\mu) \geqslant 0$, $|\arg z| < \frac{1}{2}(1-\mu)\pi$, $Re(n) > -1$, $Re(\tau) > -1$.

Taking $p = q = r = s = 1$, $a_1 = 1-\nu$, $b_1 = 0$, $\alpha_1 = \beta_1 = 1$ in (3·1) and using (2·8), we get

(4·3) $$\int_{-1}^{1} (1-x)^{\tau+\frac{m+\rho}{2}} (1+x)^{\frac{n-\sigma}{2}} P^{m,\,n}_{k-\frac{m-n}{2}}(x)\, P^{\rho,\,\sigma}_{l-\frac{\rho-\sigma}{2}}(x)\, {}_1F_0(\nu\,;\,;-z\,(1-x)^{\delta})\, dx$$

$$= \frac{2^{\tau+n+1}\,\Gamma(k+n+1)}{\Gamma(\nu)\,\Gamma(k-m+1)} \sum_{u=0}^{l} \frac{(-l)_u\,(\sigma-\rho+l+1)_u}{u\,!\,\Gamma(1-\rho+u)} \times$$

$$\times H^{2,4}_{5,5}\left[2^{\delta}z\left|\begin{matrix}(-u-\tau,\delta),(\tau-m-u,\delta),(k-\tau-m,\delta),(1-\nu,1),(-m-\tau,\delta)\\(-k-m-\tau,\delta),(0,1),(-\tau-m,\delta),(-\tau-n-k-u-1,\delta),(-\tau+k-m-u,\delta)\end{matrix}\right.\right]$$

provided $\delta > 0$; $k-\frac{m-n}{2}$, $l-\frac{\rho-r}{2}$ are positive integers, k and l are integers $\geqslant 0$, $|\arg z| < \pi$, $Re(n) > -1$, $Re(\tau) > -1$.

Taking $\alpha_j = \beta_h = 1$ ($j = 1, 2, \ldots, r$; $h = 1, 2, \ldots, s$) in (3·1) and using (2·5), we obtain

$$(4\cdot4)\quad \int_{-1}^{1}(1-x)^{\tau+\frac{m+\rho}{2}}(1+x)^{\frac{n-\sigma}{2}}P^{m,n}_{k-\frac{m-n}{2}}(x)\,P^{\rho,\sigma}_{l-\frac{\rho-\sigma}{2}}(x)\,G^{p,q}_{r,s}\left(z(1-x)^{\delta}\left|\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix}\right.\right)dx$$

$$=\frac{2^{\tau+n+1}\,\Gamma(n+k+1)}{\Gamma(k-m+1)}\sum_{u=0}^{l}\frac{(-l)_u\,(\sigma-\rho+l+1)_u}{u!\,\Gamma(1-\rho+u)}\times$$

$$\times H^{p+1,q+3}_{r+4,s+4}\left[2^{\delta}z\left|\begin{matrix}\left(-u+\left|\begin{matrix}-\tau\\\tau-m\end{matrix}\right|,\delta\right),(k-\tau-m,\delta),\{(a_r,1)\},(-m-\tau,\delta)\\(-k-m-\tau,\delta),\{(b_s,1)\},\left(-\tau+\left|\begin{matrix}-m\\-n-k-u-1\\k-m-u\end{matrix}\right|,\delta\right)\end{matrix}\right.\right]$$

where $\delta > 0$; $k-\frac{m-n}{2}$, $l-\frac{\rho-\sigma}{2}$ are positive integers, k and l are integers $\geqslant 0$, $(s-r) \geqslant 0$, $(r+s) < 2(p+q)$, $|\arg z| < (p+q-\frac{1}{2}r-\frac{1}{2}s)\pi$, $Re(n) > -1$, $Re(\tau+\delta b_j) > -1$ ($j = 1, 2, \ldots, p$).

In (3·1), putting $\alpha_j = \beta_h = 1$ ($j = 1, \ldots, r$; $h = 1, \ldots, s$); replacing q, r, s respectively by $1, q+1, p$ and setting the other parameters suitably in view of [1, p. 215(2)], we get

$$(4\cdot5)\quad \int_{-1}^{1}(1-x)^{\tau+\frac{m+\rho}{2}}(1+x)^{\frac{n-\sigma}{2}}P^{m,n}_{k-\frac{m-n}{2}}(x)\,P^{\rho,\sigma}_{l-\frac{\rho-\sigma}{2}}(x)\,E(a_1,\ldots,a_p;b_1,\ldots,b_q;z(1-x)^{\delta})dx$$

$$=\frac{2^{\tau+n+1}\,\Gamma(n+k+1)}{\Gamma(k-m+1)}\sum_{u=0}^{l}\frac{(-l)_u\,(\sigma-\rho+l+1)_u}{u!\,\Gamma(1-\rho+u)}\times$$

$$\times H^{p+1,4}_{q+5,p+4}\left[2^{\delta}z\left|\begin{matrix}\left(-u+\left|\begin{matrix}-\tau\\\tau-m\end{matrix}\right|,\delta\right),(k-\tau-m,\delta),(1,1),\{(b_q,1)\},(-m-\tau,\delta)\\(-k-m-\tau,\delta),\{(a_p,1)\},\left(-\tau+\left|\begin{matrix}-m\\-n-k-u-1\\k-m-u\end{matrix}\right|,\delta\right)\end{matrix}\right.\right]$$

where $\delta > 0$; $k-\frac{m-n}{2}$, $l-\frac{\rho-\sigma}{2}$ are positive integers, k and l are integers $\geqslant 0$,

$(p-q) \geqslant 1$, $|\arg z| < \frac{1}{2}(p-q+1)\pi$, $Re(n) > -1$, $Re(\tau + \delta a_j) > -1$ $(j = 1, \ldots, p)$.

Taking $p = s = 2$, $q = r = 0$, $b_1 = \frac{1}{2}\mu - \frac{1}{2}\nu$, $b_2 = \frac{1}{2}\mu + \frac{1}{2}\nu$, and $\beta_1 = \beta_2 = 1$, the integral (3·1) with the help of (2·5) and [1, p. 219(47)], reduces to

$$(4\cdot6)\quad \int_{-1}^{1} (1-x)^{\tau+\frac{m+\rho+\delta\mu}{2}} (1+x)^{\frac{n-\sigma}{2}} P^{m,\,n}_{k-\frac{m-n}{2}}(x)\, P^{\rho,\,\sigma}_{l-\frac{\rho-\sigma}{2}}(x)\, k_\nu\left(2z^{\frac{1}{2}}(1-x)^{\frac{\delta}{2}}\right) dx$$

$$= \frac{2^{\tau+n}\,\Gamma(k+n+1)}{\Gamma(k-m+1)}\; z^{-\frac{\mu}{2}} \sum_{u=0}^{l} \frac{(-l)_u\,(\sigma-\rho+l+1)_u}{u!\,\Gamma(1-\rho+u)} \times$$

$$\times\; H^{3,\,3}_{4,\,6}\left[2^{\delta}z \left| \begin{matrix} \left(-u+\left|\begin{matrix}-\tau\\ \tau-m\end{matrix}\right|, \delta\right), (k-\tau-m, \delta), (-m-\tau, \delta) \\ (-k-m-\tau, \delta), (\frac{1}{2}\mu-\frac{1}{2}\nu, 1), (\frac{1}{2}\mu+\frac{1}{2}\nu, 1), \left(-\tau+\left|\begin{matrix}-m\\ -n-k-u-1\\ k-m-u\end{matrix}\right|, \delta\right) \end{matrix}\right.\right]$$

where $\delta > 0$; $k - \frac{m-n}{2}$, $l - \frac{\rho-\sigma}{2}$ are positive integers, k and l are integers $\geqslant 0$, $|\arg z| < \pi$, $Re(n) > -1$, $Re(\tau + \delta(\frac{1}{2}\mu \pm \frac{1}{2}\nu)) > -1$.

Putting $p = s = 2$, $q = 0$, $r = 1$, $a_1 = a - \lambda + 1$, $b_1 = a + \mu + \frac{1}{2}$, $b_2 = a - \mu + \frac{1}{2}$ and $\alpha_1 = \beta_1 = \beta_2 = 1$ and using (2·5) and [1, p. 221(68)], the integral (3·1) reduces to

$$(4\cdot7)\int_{-1}^{1} (1-x)^{\tau+\delta a+\frac{m+\rho}{2}} (1+x)^{\frac{n+\sigma}{2}} P^{m,\,n}_{k-\frac{m-n}{2}}(x)\, P^{\rho,\,\sigma}_{l-\frac{\rho-\sigma}{2}}(x)\, e^{-\frac{1}{2}z(1-x)^{\delta}} W_{\lambda,\mu}(z(1-x)^{\delta})dx$$

$$= \frac{2^{\tau+n+1}\,\Gamma(k+n+1)\, z^{-a}}{\Gamma(k-m+1)} \sum_{u=0}^{l} \frac{(-l)_u\,(\sigma-\rho+l+1)_u}{u!\,\Gamma(1-\rho+u)} \times$$

$$\times\; H^{3,\,3}_{5,\,6}\left[2^{\delta}z \left| \begin{matrix} \left(-u+\left|\begin{matrix}-\tau\\ \tau-m\end{matrix}\right|, \delta\right), (k-\tau-m, \delta), (a-\lambda+1, 1), (-m-\tau, \delta) \\ (-k-m-\tau, \delta), (a+\mu+\frac{1}{2}, 1), (a-\mu+\frac{1}{2}, 1), \left(-\tau+\left|\begin{matrix}-m\\ -n-k-u-1\\ k-m-u\end{matrix}\right|, \delta\right) \end{matrix}\right.\right]$$

where $\delta > 0$; $k - \frac{m-n}{2}$, $l - \frac{\rho-\sigma}{2}$ are positive integers, k and l are integers $\geqslant 0$,

$|\arg z| < \frac{\pi}{2}$, $Re(n) > -1$, $Re(\tau + \delta(a \pm \mu + \frac{1}{2})) > -1$.

It may be noted that the generalised Legendre associated function reduces to associated Legendre function on setting $m = n$ and to Legendre function on putting $m = n = 0$, therefore many integrals involving such functions can be obtained as particular cases of the integrals established in section 3.

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Liquid solid countercurrent distribution of fatty acids of Brassica oil with Urea. Part I *Brassica compestris* variety *YS Pb. 24*

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Abstract

The fatty acids composition of *Brassica campestris* variety YS Pb. 24, evolved at Gurgaon sub-station of the Punjab Agricutural University, has been found by liquid solid counter current distribution with urea. The main components of the fatty acids are palmatic 0·66, stearic 1·71, behenic 3·31, lignoceric 1·32, erucic 50·6, oleic 20·49, linoleic 15·58 and linolenic acid 6·29 percent. The fatty acids of each fraction were confirmed by reverse phase chromatography.

Introduction

Brassica oil is known for the complexity of its fatty acid composition. Among the saturated fatty acids it contains palmatic, stearic, behenic and lignoceric acids. Erucic acid is the major unsaturated component.

According to Crawford and Hilditch, the higher saturated fatty acids are difficult to estimate individually, as it is not certain whether they form binary mixtures in all fractions. Use of urea complexes in the segregation of mixture of higher fatty acids has been reviewed by Schlenk and Holman[9] and by Newey *et al.*[8]. Swern and Parker[11] have described its application to the isolation of oleic, linoleic and concentrates of linolenic and linoleic acids. Recently Summerwell[10] undertook a liquid solid countercurrent distribution of fatty acids with urea. The adduct forming property of fatty acids with urea was also used for finding out the composition of various oils and fats by Mehta *et al.* [6,7].

In the present investigation, fatty acids composition of the oil of variety YS Pb. 24, evolved at Gurgaon sub-station of the Punjab Agricultural University was taken as this has not been taken up earlier and Summerwell's method was employed to fractionate the fatty acids in conjunction with Reverse Phase Chromatography.

Method and Material

Oil of *Brassica compestris* variety Yellow Sarson Pb. 24 was saponified and mixed fatty acids were obtained after removing the unsaponifiable matter. The distribution was carried out in a series of 500 ml erlenmeyer glass stoppered flasks. For fractionation mixed fatty acids 18·1 gm were dissolved in 200 ml of methanol in flask number 0, and 10 gm urea was added to it. The flask was warmed to dissolve urea, kept overnight for adduct formation. A series of conical flasks numbered 1 to 6 were charged with 10 gm of dry urea. Sufficient saturated solution of urea in a mixture of methyl alcohol (70 percent) and ethyl acetate (30

TABLE 1

Showing the composition of mixed fatty acids in the Urea–adduct fractions

Fraction	Wt. of Fatty acids gm.	Iodine value	Neutralisation equivalent	Saturated acids (gms)	Unsaturated acids (gms)	Saturated acids: Palmitic	Stearic	Behenic	Lignoceric	Unsaturated acids: Erucic	Oleic	Linoleic	Linolenic.	Fatty acids identified chromatographically
0	2·25	57·1	341·2	0·54	1·71	–	–	0·30	0·24	1·71	–	–	–	Be, lig, eru.
1.	3·53	63·8	334·0	0·53	3·00	–	0·23	0·30	–	3·00	–	–	–	St, Be, eru.
2.	1·85	66·9	328·6	0·20	1·65	0·12	0·08	–	–	1·65	–	–	–	pal, St, eru.
3.	2·80	75·0	337·9	–	2·80	–	–	–	–	2·80	–	–	–	eru.
4.	3·37	92·4	281·6	–	3·37	–	–	–	–	–	2·69	0·68	–	ol, lino.
5.	1·47	138·7	280·8	–	1·47	–	–	–	–	–	0·59	0·88	–	ol, lino.
6.	1·07	117·1	280·8	–	1·07	–	–	–	–	–	0·43	0·64	–	ol, lino,
7.	1·76	265·0	278·3	–	1·76	–	–	–	–	–	–	0·62	1·14	Lino, len.
Total	18·1			1·27	16·83	0·12	0·31	0·60	0·24	9·16	3·71	2·82	1·14	
Percentage.				7·00	92·96	0·66	1·71	3·31	1·32	50·60	20·49	15·58	6·29	

The following abbreviations have been used.

St. for stearic acid
Pal. for palmitic
Be. for behenic
lig. for lignoceric
Eru. for erucic
ol. for oleic
lino. for linoleic
len. for linolenic.

TABLE 2

Showing the variation in fatty acids composition of Brassica oil as studied by different authors

	Source	Saturated % weight: C_{14}	C_{16}	C_{18}	C_{20}	C_{22}	C_{24}	C_{26}	Unsaturated % weight: Oleic	Linoleic	Linolenic	C_{20}	Erucic
Brassica alba (4)	India	0·4	1·5	0·4	0·5	2·0	1·0	–	22·0	14·2	6·8	7·0	44·2
Brassica campestris (2)	India	–	2·0	–	5·5	–	–	1·5	12·5	16·0	8·5	5·0	47·5
,, (2)	India	–	2·0	–	7·5	–	–	2·5	16·5	13·5	7·0	5·0	45·0
,, (12)	Japan	–	4·0	–	–	1.0	–	–	14·0	24·0	2·0	–	55·0
,, *	India	–	0·7	1·7	3·3	–	1·3	–	20·5	15·6	6·3	–	50·6

*Brassica compestris variety YS Pb. 24 studied by the author.

percent) was prepared. The supernatent liquid from flask 0 was decanted into flask 1 and saturated urea solution 200 ml was added to flask number 0. Both the flasks were warmed to dissolve the contents were kept overnight for adduct formation. The process of addition of saturated urea solution to flask number 0, and warming of all the solutions in different flasks and transfer of the supernatent solution to the next flask was repeated. The quantity of solvent in flask number 6 was reduced by distillation under reduced pressure. All the fractions were treated with warm acidulated distilled water followed by extraction with ether to obtain fatty acids. The fractions were analysed for Iodine and neutralisation value by A. O. A. C. methods[1]. The confirmation of fatty acids of each fraction was made by Reverse Phase Chromatography[5]. The composition of all fractions was calculated by Hilditch's method[3]. The results are recorded in table 1.

Discussion

The data presented in table 1 shows that by liquid solid counter current distribution with urea the fatty acids are uniformly distributed. Behenic acid (3·31 percent) is the major component among the saturated fatty acid and erucic acid (50·60 percent) is the major unsaturated component. The order of saturated fatty acids are behenic, stearic, lignoceric and palmitic while that of unsaturated fatty acids are erucic, oleic, linoleic and linolenic acid. The unsaturated acids were fractionated in all the flasks (0 to 7) while saturated acids were fractionated in the first three flasks (0 to 2). Erucic acid was fractionated in the first three flasks, along with saturated acids while the flask number 3 contained pure erucic acid fraction. Maximum concentration of oleic acid was obtained in flask number 4, linoleic acid in flask number 5 and linolenic acid in flask number 7.

The perusal of data in table 2 shows the variation in the fatty acids composition of Brassica oil. In the present study the fatty acids composition of *Brassica compestris* variety YS Pb 24 evolved at Gurgaon sub-station is taken, which also shows variation in the composition.

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Self reciprocal functions

By

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Abstract

This paper deals with a theorem which is a generalisation of the theorem given by Fox. With the help of this theorem certain self reciprocal functions for the H-function defined by Fox have been obtained. A number of particular cases in G-function have been also included.

The function $k(x)$ and $h(x)$ are said to form a pair of Fourier kernels if the following pair of reciprocal equation are simulteneously valid

$$g(x) = \int_0^\infty k(xy)\; f(y)\; dy \tag{1}$$

$$f(x) = \int_0^\infty h(xy)\; g(y)\; dy \tag{2}$$

The kernels are said to be symmetrical if $k(x) = h(x)$. Fox [2, p. 408] has proved that a function $H(x)$ defined by

$$H(x) = \frac{1}{2\pi i} \int_L \frac{\prod_{m=1}^{q} \Gamma(b_m + c_m s) \prod_{n=1}^{p} \Gamma(a_n - e_n s)}{\prod_{m=1}^{q} \Gamma(b_m + c_m - c_m s) \prod_{n=1}^{p} \Gamma(a_n - e_n + e_n s)}\, x^{-s} ds$$

is a symmetric Fourtier kernel.

We shall use the symbol

$$H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix} \{a_p, e_p\} \\ \{b_q, c_q\} \end{matrix} \right]$$

for the function defined by

$$H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix} (a_1, e_1), \ldots . (a_p, e_p) \\ (b_1, c_1), \ldots . (b_q, c_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - c_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + e_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + c_j s) \prod_{j=n+1}^{p} \Gamma(a_j - e_j s)}\, x^{s} ds$$

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under this symbolic notation

$$H(x) \equiv H^{q,\,p}_{2p,\,2q}\left[x \;\middle|\; \begin{matrix} \{1-a_p,\, e_p\},\, \{a_p - e_p,\, e_p\} \\ \{b_q,\, c_q\},\, \{1-b_q-c_q,\, c_q\} \end{matrix} \right] \tag{3}$$

where the notation $\{ a_p, e_p \}$ denotes the ordered set of parameters $(a_1, e_1), a_2, e_2, \ldots\ldots (a_p, e_p)$.

A function $f(x)$ satisfying the relation

$$f(x) = \int_0^\infty k(xy)\, f(y)\, dy$$

is said to be the self reciprocal function associated with the kernel $k(x)$. All symmetric kernels can be associated with self reciprocal functions and *vice versa.*

A set of conditions have been obtained by Fox which determine a self reciprocal functions for the kernel $H(x)$ which further suggest us more general theorem by transforming the kernel $H(x)$ in more general form say $k(x)$ associated with the self reciprocal functions.

In section I, under the identical set of conditions as given by Fox [2, p. 411] for $H(x)$, we shall prove a theorem for the kernel $k(x)$ and in section II, functions have been obtained which are self reciprocal for the kernel $k(x)$.

Section I

If (3) is a symmetric Fourier kernel, then it is easy to deduce that

$$k(x) \equiv n \left(\frac{m}{n}\right)^{n/2} x^{(n-1)/2}\; H^{q,\,p}_{2p,\,2q}\left[\left(\frac{m}{n}\, x\right)^{n} \;\middle|\; \begin{matrix} \{1-a_p,\, e_p\},\, \{a_p - e_p,\, e_p\} \\ \{b_q,\, c_q\},\, \{1-b_q-c_q,\, c_q\} \end{matrix} \right] \tag{4}$$

is also a symmetric Fourier kernel.

If the Mellin transform of the function mentioned in (4) be denoted by $P(s)$, then we write

$$P(s) = \frac{Q(s)}{Q(1-s)}$$

where

$$Q(s) = \frac{\left(\frac{m}{n}\right)^{1-s/2} \prod\limits_{j=1}^{q} \Gamma\left[b_j + c_j\left(\frac{s}{n} + \frac{1}{2} - \frac{1}{2n}\right)\right]}{\prod\limits_{j=1}^{p} \Gamma\left[a_j - e_j + e_j\left(\frac{s}{n} + \frac{1}{2} - \frac{1}{2n}\right)\right]} \tag{5}$$

If we write

$$k_1(x) = \int_0^\infty k(x)\, dx$$

then by virtue of Mellin inversion formula [1, p. 307], we have

$$k_1(x) = \frac{1}{2\pi i}\int_L \frac{P(s)}{1-s}\, x^{1-s}\, ds.$$

We can now state the theorem in the following form :

Theorem I

If

(*i*) $$\int_0^x f(x)\,dx = \int_0^\infty f(y)\,\frac{k_1(xy)}{y}\,dy$$

(*ii*) $$c_j > 0,\ j = 1, \ldots, q,\ e_j > 0,\ j = 1, \ldots, p \quad \text{and}$$

$$D = 2\left(\sum_{j=1}^{q} c_j - \sum_{j=1}^{p} e_j\right) > 0$$

(*iii*) $$Re(b_j) > -c_j/2,\ j = 1, \ldots, q,$$

$$Re(a_j) > e_j/2,\ j = 1, \ldots, p$$

(*iv*) $E(\tfrac{1}{2} - s)$ is an even function of s.

and

(*v*) $$Q(s)\,E(s) \in L_2(\tfrac{1}{2} - i\infty, \tfrac{1}{2} + i\infty)$$

then

$$f(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Q(s)\,E(s)\,x^{-s}\,ds. \tag{6}$$

where $Q(s)$ is given by (5).

The proof of this theorem is identical with the theorem given by Fox [2, p. 411]

Theorem I(A)

If conditions (*i*), (*ii*), (*iii*), (*iv*) of theorem I holds and

$$Q_1(s)\,E(s) \in L_2(\tfrac{1}{2} - i\infty, \tfrac{1}{2} + i\infty)$$

then

$$f(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Q_1(s)\,E(s)\,x^{-s}\,ds \tag{7}$$

where $Q_1(s) = \left(\frac{m}{n}\right)^{1-s/2} \prod_{j=1}^{q} \Gamma\left[b_j + c_j\left(\frac{s}{n} + \frac{1}{2} - \frac{1}{2n}\right)\right] \prod_{j=1}^{p} \Gamma\left[a_j - e_j\left(\frac{s}{n} + \frac{1}{2} - \frac{1}{2n}\right)\right]$

On taking $n = 1$, $m = 1$, above theorem reduces to the theorem given by Fox [2, p. 411]

Theorem I(B)

If conditions (*i*), (*ii*), (*iii*), (*iv*) of theorem I holds and

$$Q_2(s)\,E(s) \in L_2(\tfrac{1}{2} - i\infty, \tfrac{1}{2} + i\infty)$$

then

$$f(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Q_2(s)\,E(s)\,x^{-s}\,ds \tag{8}$$

where

$$Q_2(s) = \frac{\left(\frac{m}{n}\right)^{1-s/2} \prod_{j=1}^{p} \Gamma\left[a_j - e_j\left(\frac{1}{2} + \frac{s}{n} - \frac{1}{2n}\right)\right]}{\prod_{j=1}^{q} \Gamma\left[b_j + c_j - c_j\left(\frac{s}{n} + \frac{1}{2} - \frac{1}{2n}\right)\right]}$$

Section II

In this section, we shall find the function $f(x)$ which are self reciprocal functions.

$$E(s) = \frac{\prod_{j=1}^{l} \Gamma(f_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - d_j - \alpha_j s)}{\prod_{j=l+1}^{t} \Gamma(1 - f_j - \beta_j s) \prod_{j=n+1}^{r} \Gamma(d_j + \alpha_j s)}$$

which satisfies the functional equation $E(s) = E(1-s)$ when $l = n$, $r = t$, $\alpha_j = \beta_j$ and $1 - f_j - \beta_j = d_j, j = 1, \ldots . r$.

Then we obtain by theorem I, a self reciprocal function given by

$$f(x) = \frac{1}{2\pi i}\left(\frac{m}{n}\right)\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\prod_{j=1}^{q} \Gamma(b_j + \frac{c_j}{2} - \frac{c_j}{2n} + \frac{1}{n} c_j s)}{\prod_{j=1}^{p} \Gamma(a_j - \frac{1}{2} e_j - \frac{1}{2n} e_j + \frac{1}{n} e_j s)}$$

$$\frac{\prod_{j=1}^{l} \Gamma(1 - d_j - \alpha_j + \alpha_j s)}{\prod_{j=l+1}^{r} \Gamma(d_j + \alpha_j - \alpha_j s)} \frac{\prod_{j=1}^{l} \Gamma(1 - d_j - \alpha_j s)}{\prod_{j=l+1}^{r} \Gamma(d_j + \alpha_j s)} \left[\left(\frac{m}{n}\right)^{\frac{1}{2}} x\right]^{-s} ds$$

Hence the function

$$\left(\frac{m}{n}\right) H^{l+q,\, l}_{r+p,\, r+q}\left[\left(\frac{m}{n}\right)^{\frac{1}{2}} x \,\middle|\, \begin{matrix} \{d_r, \alpha_r\}, \left\{a_p - e_p\left(\frac{1}{2} + \frac{1}{2n}\right), \frac{e_p}{n}\right\} \\ \left\{b_q, + c_q\left(\frac{1}{2} - \frac{1}{2n}\right), \frac{c_q}{n}\right\}, \{1 - d_r - \alpha_r, \alpha_r\} \end{matrix}\right] \tag{9}$$

is a self reciprocal function for the kernal mentioned in (4).

If we replace a_p, e_p, c_q by $1 - a_p$, $\frac{1}{2}e_p$, $\frac{1}{2}c_q$, take $m = \frac{1}{2}$ and $n = 1$, then by virtue of the property

$$H^{m,\, n}_{p,\, q}\left[x \,\middle|\, \begin{matrix} \{a_p, \alpha_p\} \\ \{b_q, c_q\} \end{matrix}\right] = k\, H^{m,\, n}_{p,\, q}\left[x^k \,\middle|\, \begin{matrix} \{a_p, k\ \alpha_p\} \\ \{b_q, k\ c_q\} \end{matrix}\right]$$

we get a self reciprocal functlon

$$H^{l+q,\, l}_{r+p,\, r+q}\left[\frac{x^2}{2} \,\middle|\, \begin{matrix} \{d_r, \alpha_r\}, \{1 - a_p - \frac{1}{2}e_p, e_p\} \\ \{b_q, c_q\}, \{1 - d_r - \frac{1}{2}\alpha_r, \alpha_r\} \end{matrix}\right] \tag{10}$$

for the kernel

$$\sqrt{2}\, H^{q,\, p}_{2p,\, 2q}\left[\frac{x^2}{4} \,\middle|\, \begin{matrix} \{a_p, e_p\}, \{1 - a_p - \frac{1}{2}e_p, e_p\} \\ \{b_q, c_q\}, \{1 - b_q - \frac{1}{2}c_q, c_q\} \end{matrix}\right] \tag{11}$$

On taking α^s, c^s and e^s equal to unity, we get a self reciprocal function given by Sharma [3, p. 118]

$$G^{\,l+q,\,l}_{\,r+p,\,r+q}\left[\frac{x^2}{2}\,\middle|\,\begin{matrix} d_1,\dots d_r, \frac{1}{2}-a_1,\dots \frac{1}{2}-a_p \\ b_1,\dots b_q, \frac{1}{2}-d_1,\dots \frac{1}{2}-d_r\end{matrix}\right]$$

for the kernel

$$\sqrt{2}\; G^{\,q,\,p}_{\,2p,\,2q}\left[\frac{x^2}{4}\,\middle|\,\begin{matrix} a_1,\dots a_p, \frac{1}{2}-a_1,\dots\dots, \frac{1}{2}-a_p \\ b_1,\dots b_q, \frac{1}{2}-b_1,\dots\dots, \frac{1}{2}-b_q\end{matrix}\right]$$

Similarly, with the help of Theorem I(A) and I(B), we get other self reciprocal functions given by

$$\left(\frac{m}{n}\right) H^{\,l+q,\,l+p}_{\,r+p,\,r+q}\left[\left(\frac{m}{n}\right)^{\frac{1}{2}} x\,\middle|\,\begin{matrix} \left\{1-a_p+e_p\left(\frac{1}{2}-\frac{1}{2n}\right), \frac{e_p}{n}\right\}, \{d_r, \alpha_r\} \\ \left\{b_q+c_q\left(\frac{1}{2}-\frac{1}{2n}\right), \frac{c_q}{n}\right\}, \{1-d_r-\alpha_r, \alpha_r\}\end{matrix}\right] \tag{12}$$

and

$$\left(\frac{m}{n}\right) H^{\,l,\,l+p}_{\,r+p,\,r+q}\left[\left(\frac{m}{n}\right)^{\frac{1}{2}} x\,\middle|\,\begin{matrix} \left\{1-a_p+e_p\left(\frac{1}{2}-\frac{1}{2n}\right), \frac{e_p}{n}\right\} \{d_r, \alpha_r\} \\ \left\{1-b_q-c_q\left(\frac{1}{2}+\frac{1}{2n}\right), \frac{c_q}{n}\right\}, \{1-d_r-\alpha_r, \alpha_r\}\end{matrix}\right] \tag{13}$$

for the kernel mentioned in (4).

On taking n, $e's$, $\alpha's$ and $c's$ equal to unity in the relation (9), (12) and (13), we get the self reciprocal functions given by

$$m\, G^{\,l+q,\,l}_{\,r+p,\,r+q}\left[m^{\frac{1}{2}} x\,\middle|\,\begin{matrix} d_1,\dots d_r, a_1-1,\dots a_p-1 \\ b_1,\dots b_q, -d_1,\dots\dots -d_r\end{matrix}\right] \tag{14}$$

$$m\, G^{\,l+q,\,l+p}_{\,r+p,\,r+q}\left[m^{\frac{1}{2}} x\,\middle|\,\begin{matrix} 1-a_1,\dots 1-a_p, d_1,\dots d_r \\ b_1,\dots b_q, \; d_1,\dots\dots -d_r\end{matrix}\right] \tag{15}$$

and

$$m\, G^{\,l,\,l+p}_{\,r+p,\,r+q}\left[m^{\frac{1}{2}} x\,\middle|\,\begin{matrix} 1-a_1,\dots 1-a_p, d_1,\dots\dots d_r \\ b_1,\dots -b_q, -d_1,\dots\dots -d_r\end{matrix}\right] \tag{16}$$

for the kernal

$$m^{\frac{1}{2}}\, G^{\,q,\,p}_{\,2p,\,2q}\left[m\, x\,\middle|\,\begin{matrix} 1-a_1,\dots 1-a_p, a_1-1,\dots\dots a_p-1 \\ b_1,\dots\dots b_q, -b_1,\dots\dots -b_q\end{matrix}\right] \tag{17}$$

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Variety of Problems of Neutral functional differential equations

By

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Abstract

We extend the comparison technique, comparing the solutions of a system with those of a scalar equation and obtain many results for Neutral Functional Differential systems. By this we also extend the results of Fred Brauer and Laugenhop.

Many results have been obtained by comparing the solutions of an ordinary differential system or of a functional differential system with those of a corresponding scalar differential equation. We extend this comparison technique to study various properties of the solutions of Neutral Functional differential equations. In this paper we consider the equation

(1) $$x'(t) = f(t, x(t), x(g(t)), x'(h(t)))$$

for $t > t_0$ with $x(t) = \varphi(t)$ on $[\alpha, t_0]$

where $x = (x_1, x_2, \ldots, x_n)$

$$f = (f_1, f_2, \ldots f_n)\ ;\ g = (g_1, g_2, \ldots g_m)$$

$h = (h_1, h_2, \ldots h_p)$. For $\alpha < t_0$

$$\alpha \leqq g_j(t) \leqq t \quad (j = 1, 2, \ldots m)$$

$$\alpha \leqq h_k(t) \leqq t \quad (k = 1, 2, \ldots p)$$

for all $t \geqq t_0$ and $x'(h_k(t)) = d\,\dfrac{x(h_k(t))}{dt}$. The word "Neutral" is used because the equation could be considered both for retarded arguments as well as for advanced arguments.

Suppose f, g and h are continuous function of their arguments in their appropriate domains that is f is continuous in t, $x(t)$, $x(g(t))$, $x'(h(t))$ in the domain $D = D^{1+n+nm+np}$ (an open connected set) in $E^{1+n+nm+np}$, an $(1 + n + nm + np)$ dimensional Euclidean space. Let D' be the projection of D on to the t – axis. A solution $z(t)$ of (1) may be defined if it satisfies the following conditions :

(i) $$(t, x(t), x(g(t)))\ \varepsilon\ D - F$$

for each compact set $F \subset D$, $t_0 \leqq t < \beta$

(*ii*) $x(t) = \varphi(t) \qquad \alpha \leq t \leq t_0$

(*iii*) $x(t) = f(t, x(t), x(g(t), x'(h(t)))$

for almost all $t \in (t_0, \beta)$

If $x(t)$ and $y(t)$ are any two solutions of (1) defined on an interval $[\alpha, \beta]$ we use the following metric in our paper

$$\| x(t) - g(t) \| = \operatorname*{Sup}_{\alpha \leq s \leq \beta} | x(s) - y(s) | + \int_{\alpha}^{\beta} | x'(s) - y'(s) | \, ds$$

The existence and uniqueness of Neutral Functional Differential Equations have been discussed by many authors. R. D. Driver has given references of many results obtained by various authors in this direction[1]. Our present work is inspired by the results of Driver in[1].

Theorem 1 : Suppose $w(t, r)$ is a nonnegative continuous function defined on $t \in J = [0, \infty)$ and $r \geq 0$. Suppose $f(t, x, x, z)$ satisfies the inequality.

(2)
$$\begin{aligned} \| x - y + \lambda \, [f(t, x(t), x(g(t)), x'(h(t))) \\ - f(t, y(t), y(g(t)), y'(h(t)))] \| \\ \leq \| x - y \| + \lambda \, w(t, \| x - y \|) \end{aligned}$$

for sufficiently small $h < 0$. Let $x(t)$ and $y(t)$ be any two solutions of (1) with their initial functions $\varphi(t)$ and $\psi(t)$ respectively on $[\alpha, t_0]$.

Let $r(t)$ be the maximal solution of

(3)
$$r' = w(t, r)$$

with $r(t_0) = r_0$, existing for all $t \geq t_0$.

(4) Then $\| x(t) - y(t) \| \leq r(t) \quad (t \geq t_0)$

(5) provided $\| \phi - \psi \| \leq r_0$.

Proof of theorem 1 : Let $m(t) = \| x(t) - y(t) \|$ where $x(t)$ and $y(t)$ are any two solutions of (1) with their initial functions $\phi(t)$ and $\psi(t)$ such that

$$\| \phi - \psi \| \leq r_0$$

$$\begin{aligned} m(t + \lambda) &= \| x(t + \lambda) - y(t + \lambda) \| \\ &\leq \| x(t) - y(t) + \lambda \, [f(t, x(t), x(g(t)), x'(h(t))) \\ &\quad - f(t, y(t), y(g(t)), y'(h(t)))] \| \\ &\quad + \varepsilon_1 \lambda + \varepsilon_2 \lambda \end{aligned}$$

where ε_1 and ε_2 tend to zero as $\lambda \to 0$ using the inequality (2) we obtain

(6) $$\limsup_{\lambda \to 0+} \frac{[m(t+\lambda) - m(t)]}{\lambda} \leqq w(t, m(t))$$

Let $r_\varepsilon(t)$ be any solution of

$$r'(t) = w(t, r) + \varepsilon$$
$$r(t_0) = r_0$$

where $\varepsilon > 0$ is a sufficiently small number.

To show

(7) $$m(t) \leqq r_\varepsilon(t) \qquad \text{for } t \geqq t_0$$

If the inequality (7) does not hold suppose there exists a point ζ to the right of t_0 such that $m(\zeta) = r_\varepsilon(\zeta)$ and that $m(t) > r_e(t)$ for $t > \zeta$ and sufficiently near ζ. At such a ζ the right hand derivative $m'_+(\zeta)$ exists and

(8) $$\limsup_{\lambda \to 0+} \frac{[\,m(\zeta + h) - m(\zeta)\,]}{\lambda} \geqq r'_\varepsilon(\zeta)$$

$$= w(\zeta, r_\varepsilon(\zeta)) + \varepsilon$$

which contradicts (6)

Hence the inequality (ζ) holds

$$\lim_{\varepsilon \to 0} r_\varepsilon(t) = r(t)$$

Now the result follows.

Theorem 2 : Let the assumptions in theorem 1 hold. Suppose the maximal solution

$$r' = w(t, r)$$
$$\text{with } r(t_0) = r_0 = 0$$

is identically zero.

Then there is at most one solution of (1) with the initial functions $\phi = \psi$ at $t = t_0$.

Proof of theorem 2 is analogous to that of theorem 1.

Corollary 1 : Suppose $w(t, r) = k\,r + \varepsilon$ where k is a constant and $\varepsilon > 0$ is sufficiently small then the inequality (4) in theorem 1 reduces to

$$\| x(t) - y(t) \| \leqq r_0\, e^{k(t-t_0)} + \frac{\varepsilon}{k}\,(e^{k(t-t_0)} - 1)$$

this extends the results of Fred Brauer to Neutral Functional Differential Equations.

Corollary 2 : Suppose $w(t, r) = v(t)\, g(r)$ where $g(r)$ is a non-negative continuous function. If $G(r) = \int_{r_0}^{r} \frac{dr}{g(r)}$ and G^{-1} is the inverse function of G, then we can easily prove

$$\| x(t) - y(t) \| \leq G^{-1} \left[G(r_0) + \int_{t_0}^{t} v(t)\, dt \right]$$

as long as $G(r_0) + \int_{t_0}^{t} v(s)\, ds$ lies in the domain of G^{-1} for all $t \geq t_0$

This extends the results of Laugenhop to Neutral Functional Differential Equations.

Remark 1. Many properties including the stability and boundedness properties defined with respect to a clock space studied by J. L. Massera[5] satisfied by a scalar differential equation imply the corresponding properties satisfied by the Neutral Functional Differential Equation (1).

Remark 2. If $y(t)$ instead of being a solution of (1), is an element of a non-empty set s, then we can study the stability and boundedness of a set with respect to (1). This extends the results of Yoshizawa[6].

Remark 3. The above results can be extended to the perturbed equation.

$$\dot{x}(t) = f(t, x(t), x(g(t)), \dot{x}(h(t)))$$
$$+ F(t, x(t), x(g(t)), \dot{x}(h(t)))$$

where F is a perturbed function and obtain similar properties under constantly acting perturbations.

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On Fox H-Transform in two Variables

By

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Abstract

The Laplace Transform in two variables is defined by the convergent double integral

$$\phi(p, q) = p\, q \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\, dx\, dy, \qquad R(p, q) > 0.$$

In the present paper, the author introduces a generalization of this in the form

$$\phi(p, q) = \lambda\, \mu\, p\, q \int_0^\infty \int_0^\infty H^{m,\, o}_{n,\, m}\left[c\, p\, x \,\middle|\, \begin{matrix}(a_n, c)\\ (b_m, c)\end{matrix}\right] H^{\alpha,\, o}_{\beta,\, \alpha}\left[d\, q\, y \,\middle|\, \begin{matrix}(h_\beta, d)\\ (f\alpha, d)\end{matrix}\right] \times$$
$$\times f(x, y)\, dx\, dy$$

where

$$H^{k,\, l}_{r,\, t}\left[x \,\middle|\, \begin{matrix}(u_r, e_r)\\ (v_t, c_t)\end{matrix}\right] \equiv H^{k,\, l}_{r,\, t}\left[x \,\middle|\, \begin{matrix}(u_1, e_1), \ldots, (u_r, e_r)\\ (v_1, c_1), \ldots, (v_t, c_t)\end{matrix}\right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{k} \Gamma(v_j - c_j.s) \prod_{j=1}^{l} \Gamma(1 - u_j + e_j.s)}{\prod_{j=k+1}^{t} \Gamma(1 - v_j + c_j.s) \prod_{j=l+1}^{r} \Gamma(u_j - e_j.s)} \,.\, x^s\, ds$$

is the H–Function of Fox which is more general than even the G–Function of Meijer.

Enunciating some fundamental theorems, the author obtains images of a few functions in this transform and further establishes a theorem which together with the other results obtained offers a set of most general results from which, by specializing the parameters involved, almost all the results of Rathie published in a recent paper may be deduced as particular cases.

1. Introduction : Fox [4, p. 408(52)] has defined a function

$$(1\cdot1) \qquad H(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{m=1}^{q} \Gamma(b_m + c_m.s) \prod_{n=1}^{p} \Gamma(a_n - e_n.s)}{\prod_{m=1}^{q} \Gamma(b_m + c_m - c_m.s) \prod_{n=1}^{p} \Gamma(a_n - e_n + e_n.s)}\, x^{-s}\, ds$$

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which is more general than Meijer's G-Function. He has further proved that it is a Symmetric Fourier Kernel. Recently Verma [7] has shown that the kernel plays the role of a transform and has defined

$$F(p) = \int_0^\infty H(px) f(x)\, dx$$

as the Fox–Transform of $f(x)$ when the integral on the right converges.

In the present paper we introduce this function to generalize the well-known Laplace Transform in two variables :

(1·2) $$\phi(p, q) = p\, q \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\, dx\, dy, \qquad R(p, q) > 0,$$

which, by analogy with the notation of operational calculus in one variable we denote as

$$\phi(p, q) \doteqdot f(x, y)$$

and put the generalization as :

(1·3) $$\phi(p, q) = \lambda\, \mu\, p\, q \int_0^\infty \int_0^\infty H^{m,\,0}_{n,\,m}\left[c\, p\, x \,\middle|\, \begin{matrix} (a_n, c) \\ (b_m, c) \end{matrix} \right] H^{\alpha,\,0}_{\beta,\,\alpha}\left[d\, q\, y \,\middle|\, \begin{matrix} (h_\beta, d) \\ (f_\alpha, d) \end{matrix} \right] \times f(x, y)\, dx\, dy$$

where $m > n$, $|\arg p| < \dfrac{m-n}{2}\,\pi\, c$, $\alpha > \beta$,

$|\arg q| < \dfrac{\alpha-\beta}{2}\,\pi\, d$; c, d are positive real constants,

$$f(x, y) = O(x^{l_1}) \text{ for small } x,$$
$$= O(y^{l_2}) \text{ for small } y,$$

$$R\left(l_1 + \frac{b_j}{c} + 1\right) > 0,\ R\left(l_2 + \frac{f_i}{d} + 1\right) > 0,\ j = 1, 2, 3, \ldots, m\,;$$

$i = 1, 2, 3, \ldots, \alpha$.

Here, following Gupta [5, p. 98(4)], we define the H-Function in the form :

$$H^{k,\,l}_{r,\,t}\left[x \,\middle|\, \begin{matrix} (u_r, e_r) \\ (v_t, c_t) \end{matrix} \right] = H^{k,\,l}_{r,\,t}\left[x \,\middle|\, \begin{matrix} (u_1, e_1), \ldots, (u_r, e_r) \\ (v_1, c_1), \ldots, (v_t, c_t) \end{matrix} \right]$$

(1·4) $$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{k} \Gamma(v_j - c_j.s) \prod_{j=1}^{l} \Gamma(1 - u_j + e_j.s)}{\prod_{j=k+1}^{t} \Gamma(1 - v_j + c_j.s) \prod_{j=l+1}^{r} \Gamma(u_j - e_j.s)}\, x^s\, ds$$

we shall represent (1·3) symbolically by

$$\phi(p, q) \;\overset{H}{\underset{m,\, n\, ;\, \alpha,\, \beta}{=\!=\!=}}\; f(x, y)$$

we shall also use the notation

$$\phi(p) \xlongequal[m,\, n]{H} f(x)$$

to denote the transform

$$(1\cdot5) \qquad \phi(p) = \lambda\, p \int_0^\infty H\, {}^{m,\, o}_{n,\, m} \left[c\, p\, x \,\middle|\, \begin{matrix} (a_n, c) \\ (b_m, c) \end{matrix} \right] f(x)\, dx$$

It may be remarked here that when $\lambda = \mu = \dfrac{1}{2\sqrt{\pi}}$, $c = d = \frac{1}{2}$, $n = o = \beta$, $m = 2 = \alpha$, $b_1 = \frac{1}{4} + \dfrac{\nu}{2}$, $b_2 = \frac{1}{4} - \dfrac{\nu}{2}$, $f_1 = \frac{1}{4} + \dfrac{\mu}{2}$, $f_2 = \frac{1}{4} - \dfrac{\mu}{2}$,

then by virtue of the relation

$$H\, {}^{2,\, 0}_{0,\, 2} \left[x \,\middle|\, \left(\frac{l}{2} + \frac{1}{4} \pm \frac{\nu}{2}, \frac{1}{2} \right) \right] = 4.\, x^{l+\frac{1}{2}}\, K_\nu\, (2\, x)$$

(1·3) reduces to the form :

$$(1\cdot6) \quad \phi(p, q) = \frac{2}{\pi} p, q \int_0^\infty \int_0^\infty (p\, x)^{\frac{1}{2}}\, (q\, y)^{\frac{1}{2}}\, K_\nu(p\, x)\, K_\mu(q\, y)\, f(x, y)\, dx\, dy, \quad R(p, q) > 0$$

which is Meijer's Bessel Transform of two variables introduced recently by Rathie [6, p. 41(4)] and symbolically denoted by him as :

$$\phi(p, q) \xlongequal[\nu,\, \mu]{K} f(x, y)$$

The following results shall be required in the sequel :

$$(1\cdot7) \qquad x^\sigma\, H\, {}^{k,\, l}_{r,\, t} \left[x \,\middle|\, \begin{matrix} (u_r, e_r) \\ (v_t, c_t) \end{matrix} \right] = H\, {}^{k,\, l}_{r,\, t} \left[x \,\middle|\, \begin{matrix} (u_r + \sigma\, e_r, e_r) \\ (v_t + \sigma\, c_t, c_t) \end{matrix} \right].$$

$$(1\cdot8) \qquad H\, {}^{k,\, l}_{r,\, t} \left[x^{-1} \,\middle|\, \begin{matrix} (u_r, e_r) \\ (v_t, c_t) \end{matrix} \right] = H\, {}^{l,\, k}_{t,\, r} \left[x \,\middle|\, \begin{matrix} (1 - v_t, c_t) \\ (1 - u_r, e_r) \end{matrix} \right]$$

$$(1\cdot9) \qquad \int_0^\infty H\, {}^{k,\, l}_{r,\, t} \left[a\, x \,\middle|\, \begin{matrix} (u_r, e_r) \\ (v_t, c_t) \end{matrix} \right] x^{s-1}\, dx$$

$$= \frac{\prod_{j=1}^{k} \Gamma(v_j + c_j.s) \prod_{j=1}^{l} \Gamma(1 - u_j - e_j.s)}{\prod_{j=k+1}^{t} \Gamma(1 - v_j - c_j.s) \prod_{j=l+1}^{r} \Gamma(u_j + e_j.s)}\, a^{-s}$$

provided $-\min\limits_{1 \leqslant j \leqslant k} R\left(\dfrac{v_j}{c_j}\right) < R(s) < \dfrac{1}{e_j} - \max\limits_{1 \leqslant j \leqslant l} R\left(\dfrac{u_j}{e_j}\right)$

$$\gamma = \sum_{j=1}^{l} (e_j) - \sum_{j=l+1}^{r} (e_j) + \sum_{j=1}^{k} (f_j) - \sum_{j=k+1}^{t} (e_j) > 0,$$

and $| \arg (\alpha) | < \frac{1}{2} \gamma \pi$.

(1·10)
$$\int_0^\infty H^{m,\, n}_{p,\, q}\left[\alpha x^{\sigma} \,\middle|\, \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix}\right] H^{k,\, l}_{r,\, t}\left[\beta x^{\lambda} \,\middle|\, \begin{matrix} (c_r, h_r) \\ (d_t, u_t) \end{matrix}\right] dx$$

$$= \frac{1}{\sigma\, \lambda(\beta)^{1/\lambda}}\; H^{l+m,\, n+k}_{p+t,\, q+r}\left[\frac{\alpha^{1/\sigma}}{\beta^{1/\lambda}} \,\middle|\, \begin{matrix} \left(a_1, \frac{e_1}{\sigma}\right), \ldots, \left(a_n, \frac{e_n}{\sigma}\right), \left(1 - d_t - \frac{u_t}{\lambda}, \frac{u_t}{\lambda}\right) \\ \left(a_{n+1}, \frac{e_{n+1}}{\sigma}\right), \ldots, \left(a_p, \frac{e_p}{\sigma}\right) \\ \left(b_1, \frac{f_1}{\sigma}\right), \ldots, \left(b_m, \frac{f_m}{\sigma}\right), \left(1 - c_r - \frac{h_r}{\lambda}, \frac{h_r}{\lambda}\right) \\ \left(b_{m+1}, \frac{f_{m+1}}{\sigma}\right), \ldots, \left(b_q, \frac{f_q}{\sigma}\right) \end{matrix}\right]$$

where $-\sigma \min\limits_{1 \leqslant j \leqslant m} R\left(\frac{b_j}{f_j}\right) < R(s) < \frac{\sigma}{e_j} - \sigma \max\limits_{1 \leqslant j \leqslant n} R\left(\frac{a_j}{e_j}\right)$

$$\gamma = \sum_{j=1}^{n} (e_j) - \sum_{j=n+1}^{p} (e_j) + \sum_{j=1}^{m} (f_j) - \sum_{j=m+1}^{q} (f_j) > 0,$$

$$| \arg (\alpha) | < \tfrac{1}{2} \gamma \pi,$$

with the corresponding conditions involving c_j, h_j, d_j, u_j, λ, β, etc. and

$$\lambda \min_{1 \leqslant j \leqslant k} R\left(\frac{d_j}{u_j}\right) + \sigma \min_{1 \leqslant j \leqslant m} R\left(\frac{b_j}{f_j}\right) + 1 > 0,$$

$$\sigma\left[\frac{1 - \max\limits_{1 \leqslant j \leqslant n} R(a_j)}{e_j}\right] + \lambda\left[\frac{1 - \max\limits_{1 \leqslant j \leqslant l} R(c_j)}{h_j}\right] > 1.$$

The symbol $\triangle\, (n\,;\, \alpha)$ shall be used to denote the set of parameters

$$\frac{\alpha}{n}, \frac{\alpha + 1}{n}, \ldots, \frac{\alpha + n - 1}{n}.$$

2. We enunciate a few fundamental Theorems. The proofs are simple and may be omitted.

Theorem I.

(2·1) If
$$\phi(p, q) \; \frac{H}{m,\, n\,;\, \alpha,\, \beta} \; f(x, y),$$

then
$$\phi\left(\frac{p}{a}, \frac{q}{b}\right) \frac{H}{m,\, n\,;\, \alpha,\, \beta} f(ax, by)$$

Theorem II.

(2·2) If $\phi_i(p, q) \xrightarrow[m,\, n\, ;\, \alpha,\, \beta]{H} f_i(x, y),\ i = 1, 2.$

then

$$\int_0^\infty \int_0^\infty \phi_1(u, v) f_2(u, v) \frac{du\, dv}{uv} = \int_0^\infty \int_0^\infty \phi_2(u, v) f_1(u, v) \frac{du\, dv}{uv}$$

provided the integral involved are uniformly and absolutely convergent.

Theorem III.

(2·3) If $f(x, y) = f_1(x).\, (f_2(y),$

then $\phi(p, q) = \phi_1(p).\ \phi_2(q)$

where,

$$\phi_1(p) \xrightarrow[m,\, n]{H} f_2(x),\ \phi_2(p) \xrightarrow[\alpha,\, \beta]{H} f_2(x)$$

provided the integrals involved converge.

Theorem IV.

(2·4) If $\phi(p, q) \xrightarrow[m,\, n\, ;\, \alpha,\, \beta]{H} f(x, y),$

then (a) $\int_p^\infty \int_q^\infty \phi(u, v) \frac{du\, dv}{uv} \xrightarrow[m,\, n\, ;\, \alpha,\, \beta]{H} \int_0^x \int_0^y f(u, v) \frac{du\, dv}{uv}$

(b) $\int_0^p \int_0^q \phi(u, v) \frac{du\, dv}{uv} \xrightarrow[m,\, n\, ;\, \alpha,\, \beta]{H} \int_x^\infty \int_y^\infty f(u, v) \frac{du\, dv}{uv}$

(c) $\int_0^\infty \int_0^\infty \phi\, (u, v) \frac{du\, dv}{u^l\, v^\nu}.$

$$= \lambda\mu c^{\rho-2}\, d^{\nu-2} \frac{\prod_{j=1}^{m} \Gamma\{ b_j + c(2-\rho) \} \prod_{j=1}^{\alpha} \Gamma\{ f_j + d(2-\nu) \}}{\prod_{j=1}^{n} \Gamma\{ a_j + c(2-\rho) \} \prod_{j=1}^{\beta} \Gamma\{ h_j + d(2-\nu) \}} \times$$

$$\times \int_0^\infty \int_0^\infty u^{\rho-2}\, v^{\nu-2} f(u, v)\, du\, dv.$$

provided the integrals involved exist.

3. We shall utilize above theorems to obtain images of a few functions in this transform.

Example 1. Let us take $f(x, y) = x^a.\, y^b$ we have from (1·9) and Theorem III.

$$\phi(p,q) = \frac{\lambda\,\mu\,p^{-a}\,q^{-b}}{c^{a+1}\,d^{b+1}} \frac{\prod_{j=1}^{m} \Gamma\{b_j + c(a+1)\} \prod_{j=1}^{\alpha} \Gamma\{f_j + d(b+1)\}}{\prod_{j=1}^{n} \Gamma\{a_j + c(a+1)\} \prod_{j=1}^{\beta} \Gamma\{h_j + d(b+1)\}}$$

where

$$R\left(a + \frac{b_j}{c} + 1\right) > 0,\; R\left(b + \frac{f_j}{d} + 1\right) > 0,$$

$$j = 1, 2, 3, \ldots, m, \qquad i = 1, 2, 3, \ldots, \alpha.$$

along with the conditions on other parameters as given in (1·3)

So that

$$(3{\cdot}1) \qquad \frac{\lambda\,\mu\,p^{-a}\,q^{-b}}{c^{a+1}\,d^{b+1}} \cdot \frac{\prod_{j=1}^{m} \Gamma\{b_j + c(a+1)\} \prod_{j=1}^{\alpha} \Gamma\{f_j + d(b+1)\}}{\prod_{j=1}^{n} \Gamma\{a_j + c(a+1)\} \prod_{j=1}^{\beta} \Gamma\{h_j + d(b+1)\}}$$

$$\frac{H}{m, n;\, \alpha, \beta}\; x^a, y^b$$

If we put $\qquad \lambda = \frac{1}{2\sqrt{\pi}} = \mu,\; c = d = \frac{1}{2},\; n = 0 = \beta,\; m = 2 = \alpha,$

$$b_1 = \tfrac{1}{4} + \frac{\nu}{2},\; b_2 = \tfrac{1}{4} - \frac{\nu}{2},\; f_1 = \tfrac{1}{4} + \frac{\mu}{2},\; f_2 = \tfrac{1}{4} - \frac{\mu}{2}$$

we get the result [6, p. 43(8)] by Rathie.

Example 2. If we take

$$f(x, y) = x^{a_1+1} \,.\, y^{a_2+1} \,.\, e^{-ax-by}$$

then it can be easily seen from Gupta [5, p. 99(6)] that

$$\int_0^\infty x^{a_1+1} \,.\, e^{-ax} \,.\, H^{m,o}_{n,m}\left[c p x \,\middle|\, \begin{matrix}(a_n, c)\\ (b_m, c)\end{matrix} \right] dx$$

$$= \sqrt{\frac{2}{\pi}} \,.\, 2^{a_1+\frac{1}{2}} \,.\, a^{-a_1-2} \,.\, H^{m,2}_{n+2,m}\left[\frac{2cp}{a} \,\middle|\, \begin{matrix}\left(\frac{-a_1}{2}, \frac{1}{2}\right), \left(\frac{-a_1}{2} - \frac{1}{2}, \frac{1}{2}\right), (a_n, c)\\ (b_m, c)\end{matrix} \right]$$

provided $\qquad R(a) > 0,\; R\left(a_1 + \frac{3}{2} \pm \frac{1}{2} + \frac{b_j}{c}\right) > 0,\; j = 1, 2, \ldots, m > n,$

$$|\arg p| < \frac{m-n}{2}\,\pi c.$$

Hence by Theorem III,

$$\phi(p,q)=\frac{\lambda\mu\, pq\, 2^{\alpha_1+\alpha_2+2}}{\pi\, a^{\alpha_1+2}\, b^{\alpha_2+2}}\, H^{m,2}_{n+2,m}\left[\frac{2\,cp}{a}\middle|\begin{matrix}\left(\frac{-\alpha_1}{2},\frac{1}{2}\right),\left(\frac{-\alpha_1}{2}-\frac{1}{2},\frac{1}{2}\right),(a_n,c)\\(b_m,c)\end{matrix}\right]$$

$$\tag{3·2}\times H^{\alpha,2}_{\beta+2,\alpha}\left[\frac{2\,dq}{b}\middle|\begin{matrix}\left(\frac{-\alpha_2}{2},\frac{1}{2}\right),\left(\frac{-\alpha_2}{2}-\frac{1}{2},\frac{1}{2}\right)(h_\beta,d)\\(f_\alpha,d)\end{matrix}\right],$$

$$\underset{m,\,n\,;\,\alpha,\,\beta}{\overset{H}{=\!=}}\; x^{\alpha_1+1}\, y^{\alpha_2+1}\, e^{-ax-by}.$$

Putting

$$\lambda=\frac{1}{2\sqrt{\pi}}=\mu,\; c=d=\tfrac{1}{2},\; n=0=\beta,$$

$m=2=\alpha$, $b_1=\frac{1}{4}+\frac{\nu}{2}$, $b_2=\frac{1}{4}-\frac{\nu}{2}$, $f_1=\frac{1}{4}+\frac{\mu}{2}$, $f_2=\frac{1}{4}-\frac{\mu}{2}$, making use of (1·4), duplication formula for Gamma functions and [1, 2·10(1) (5), 2·11(6)], Rathie's result [6, p. 44(9)] under relevant conditions can be deduced.

Example 3. Further if

$$f(x,y)=x^{\alpha_1}y^{\alpha_2}K_{\nu_1}\left(a\,x^{\frac{-m_1}{m_2}}\right)K_{\mu_1}\left(b\,x^{\frac{-n_1}{n_2}}\right),$$

it is easy to see that

$$x^{\alpha_1}K_{\nu_1}\left(ax^{\frac{-m_1}{m_2}}\right)=2^{\frac{-\alpha_1 m_2}{m_1}-2}\cdot a^{\frac{\alpha_1 m_2}{m_1}}\, H^{2,0}_{0,2}\left[\frac{ax}{2}^{\frac{-m_1}{m_2}}\middle|\left(\frac{-\alpha_1 m_2}{2\,m_1}\pm\frac{\nu_1}{2},\frac{1}{2}\right)\right]$$

Using (1·8) and (1·10) we obtain

$$\int_0^\infty x^{\alpha_1}.\,K_{\nu_1}\left(ax^{\frac{-m_1}{m_2}}\right)H^{m,0}_{n,m}\left[cpx\middle|\begin{matrix}(a_n,c)\\(b_m,c)\end{matrix}\right]dx$$

$$=\frac{m_2}{4\,m_1}\left(\frac{a}{2}\right)^{\frac{m_2}{m_1}(\alpha_1+1)}H^{m+2,0}_{n,m+2}\left[cp\left(\frac{a}{2}\right)^{\frac{m_2}{m_1}}\middle|\begin{matrix}(a_n,c)\\(b_m,c),\left(\frac{-\alpha_1 m_2}{2m_1}-\frac{m_2}{2m_1}\mp\frac{\nu_1}{2},\frac{m_2}{2m_1}\right)\end{matrix}\right]$$

provided

$$R\left(\alpha_1+\frac{m_1}{2m_2}+\frac{b_j}{c}+1\right)>0,\; j=1,2,3,\ldots,m,\quad m>n,$$

$$|\arg p|<\frac{m-n}{2}\pi c,\; a>0.$$

which gives with the help of Theorem III,

$$\phi(p, q) = \frac{\lambda\mu\, pq\, m_2 n_2}{16\, m_1 n_1} \left(\frac{a}{2}\right)^{\frac{m_2}{m_1}(\alpha_1+1)} \left(\frac{b}{2}\right)^{\frac{n_2}{n_1}(\alpha_2+1)} \times$$

$$\times\, H^{m+2,\, 0}_{n,\, m+2}\left[c\, p\left(\frac{a}{2}\right)^{\frac{m_2}{m_1}} \middle| \begin{matrix} (a_n, c) \\ (b_m, c), \left(\frac{-\alpha_1 m_2}{2\, m_1} - \frac{m_2}{2\, m_1} \mp \frac{\nu_1}{2}, \frac{m_2}{2 m_1}\right) \end{matrix} \right] \times$$

$$\text{(3·3)} \quad \times\, H^{\alpha+2,\, 0}_{\beta,\, \alpha+2}\left[dq\left(\frac{b}{2}\right)^{\frac{n_2}{n_1}} \middle| \begin{matrix} (h_\beta, d) \\ (f_\alpha, d), \left(\frac{-\alpha_2 n_2}{2\, n_1} - \frac{n_2}{2\, n_1} \pm \frac{\mu_1}{2}, \frac{n_2}{2\, n_1}\right) \end{matrix} \right]$$

$$\frac{H}{m, n\, ;\, \alpha, \beta}\; x^{\alpha_1} \cdot y^{\alpha_2} \cdot K_{\nu_1}\left(a\, x^{\frac{m_1}{m_2}}\right) K_{\mu_1}\left(b\, x^{\frac{n_1}{n_2}}\right).$$

Specializing the parameters, with

$$\lambda = \frac{1}{2\sqrt{\pi}} = \mu, \quad m = 2 = \alpha, \quad n = 0 = \beta, \quad c = \tfrac{1}{2} = d,$$

$$b_1 = \tfrac{1}{4} + \frac{\nu}{2}, \; b_2 = \tfrac{1}{4} - \frac{\nu}{2}, \; f_1 = \tfrac{1}{4} + \frac{\mu}{2}, \; f_2 = \tfrac{1}{4} - \frac{\mu}{2},$$

we arrive at Rathie's result [6, p. 44(10)] with the help of (1·4), Gauss's multiplication formula for Gamma functions and a little simplification.

Example 4. Finally, taking

$$f(x, y) = H^{k,\, l}_{r,\, t}\left[a x^{\gamma} y^{\delta} \middle| \begin{matrix} (u_r, e_r) \\ (v_t, c_t) \end{matrix} \right]$$

using (1·10) to evaluate first the x-integral and then the y-integral we get

$$\phi(p, q) = \frac{\lambda\mu\, pq}{\delta.(a)^{1/\delta}}\, (cp)^{\frac{\gamma}{\delta}-1} \times$$

$$\times\, H^{\alpha+l+m,\, k}_{\beta+n+t,\, \alpha+m+r}\left[\frac{dq(cp)^{\frac{\gamma}{\delta}}}{(a)^{1/\delta}} \middle| \begin{matrix} \left(1-v_t-\frac{c_t}{\delta}, \frac{c_t}{\delta}\right), \left(a_n + c - \frac{\gamma c}{\delta}, \frac{\gamma c}{\delta}\right), (h_\beta, d), \\ (f_\alpha, d), \left(b_m + c - \frac{\gamma c}{\delta}, \frac{\gamma c}{\delta}\right)\left(1 - u_r - \frac{e_r}{\delta}, \frac{e_r}{\delta}\right) \end{matrix} \right]$$

$$\frac{H}{m, n\, ;\, \alpha, \beta}\; H^{k,\, l}_{r,\, t}\left[a x^{\gamma} y^{\delta} \middle| \begin{matrix} (u_r, e_r) \\ (v_t, c_t) \end{matrix} \right]$$

provided $\quad \min\limits_{1 \leq j \leq m} R\left(\frac{b_j}{c}\right) + \gamma \min\limits_{1 \leq j \leq k} R\left(\frac{v_j}{c_j}\right) + 1 > 0,$

$$\min_{1 \leqslant j \leqslant \alpha} R\left(\frac{f_j}{d}\right) + \delta \min_{1 \leqslant j \leqslant k} R\left(\frac{v_j}{c_j}\right) + 1 > 0,$$

$$\rho = \sum_{j=1}^{l} (e_j) - \sum_{j=l+1}^{r} (e_j) + \sum_{j=1}^{k} (c_j) - \sum_{j=k+1}^{t} (c_j) > 0.$$

Particular cases :

Case I. If $\lambda = \mu = 1,\ m = 1 = \alpha,\ n = 0 = \beta,\ b_1 = 0 = f_1,$
$c = 1 = d,\ \gamma = \delta = 1,\ k = 1,\ l = 0 = r,\ t = 2,$
$v_1 = v - 1,\ c_1 = 1,\ v_2 = 0,\ c_2 = 1,$

H–Function degenerates and we get

$$\frac{\Gamma(\nu) \cdot pq}{(pq+a)^{\nu}} \doteq \left(\frac{xy}{a}\right)^{\frac{\nu-1}{2}} J_{\nu-1}(2\sqrt{axy}), \qquad R(\nu) > 0.$$

which is known [3, p. 135(2·74)].

Case II. Substituting $\frac{1}{2} p^{-\lambda} q^{-\nu}$ for $\lambda\mu$ and putting $c = d = 1 = a = m = \alpha$, $n = 0 = \beta = l = r,\ k = 1,\ t = 2,\ \gamma = \frac{\mu}{2},\ \delta = \frac{1}{2},\ b_1 = \lambda,\ f_1 = \nu,\ v_1 = 0,\ c_1 = \frac{1}{2},$ $v_2 = -\lambda,\ c_2 = \frac{\mu}{2}$, we get

$$\phi(p, q) = \frac{p^{1-\lambda} q^{1-\nu}}{2} \int_0^\infty \int_0^\infty H_{0,1}^{1,0}[px \mid (\lambda, 1)]\, H_{0,1}^{1,0}[qy \mid (\nu, 1)] \times$$
$$\times H_{0,1}^{1,0}\left[x^{\frac{\mu}{2}} y^{\frac{1}{2}} \,\middle|\, (0, \tfrac{1}{2}), \left(-\lambda, \frac{\mu}{2}\right)\right] dx\, dy$$

$$= p^{\mu-\lambda} q^{1-\nu} H_{1,1}^{1,1}\left[q p^{\mu} \,\begin{matrix}(0, 1)\\(\nu, 1)\end{matrix}\right] = \frac{\Gamma(\nu+1)\, q \cdot p^{\mu(\nu+1)-\lambda}}{(1 + q p^{\mu})^{\nu+1}}, \quad R(\nu + 1) > 0$$

and we obtain the known result [3, p. 140(2·113)], *viz.*

$$\frac{\Gamma(\nu+1)\, q\, p^{\mu(\nu+1)-\lambda}}{(1 + p^{\mu}.q)^{\nu+1}} \doteq x^{\lambda} y^{\nu} J_{\lambda}^{\mu}(x^{\mu}.y)$$

Case III. If we take $\frac{1}{2} p^{\mu-\lambda} q^{1-\nu}$ for $\lambda\mu$ and $m = \alpha = c = d = 1$, $a = \frac{1}{\sqrt{2}},\ n = 0 = \beta = l = r,\ k = 1,\ t = 2,\ \gamma = \mu,\ \delta = 1,\ b_1 = \lambda - \mu,\ f_1 = \nu - 1,$ $v_1 = 0,\ c_1 = \frac{1}{2},\ v_2 = \mu - \lambda,\ c_2 = \mu$, we get

$$\phi(p, q) = \tfrac{1}{2} p^{1+\mu-\lambda} . q^{2-\nu} \int_0^\infty \int_0^\infty H_{0,1}^{1,0}[px \mid (\lambda - \mu, 1)]\, H_{0,1}^{1,0}[qy \mid (\nu - 1, 1)] \times$$
$$\times H_{0,2}^{1,0}\left[\frac{yx^{\mu}}{\sqrt{2}} \,\middle|\, (0, \tfrac{1}{2}), (\mu - \lambda, \mu)\right] dx\, dy$$

$$= \frac{2^{\nu-\frac{5}{2}}}{\sqrt{\pi}} p^{2\mu-\lambda} . q^{2-\nu} . H\,{}^{2,\,1}_{1,\,2}\left[\frac{p^{\mu}.q}{\sqrt{2}}\middle|\begin{matrix}(\frac{1}{2},\frac{1}{2})\\ \left(\frac{\nu-1}{2},\frac{1}{2}\right)\left(\frac{\nu}{2},\frac{1}{2}\right)\end{matrix}\right]$$

which can be easily proved to be equal to

$$\Gamma(\nu)\, p^{\mu(1+\nu)-\lambda} . q\, e^{\frac{1}{4} p^{2\mu}.q^2} . D_{-\nu}(p^{\mu}.q)$$

with the help of [2, p. 117(3)] and [1, p. 221(69)], $D_{-\nu}(x)$ being the parabolic cylinder function.

Hence $\Gamma(\nu)\ p^{\mu(1+\nu)-\lambda} . q\, e^{\frac{1}{4} p^{2\mu}. q^2}\ D_{-\nu}(p^{\mu}.q)$

$$\doteqdot x^{\lambda-\mu} . y^{\nu-1}\ J^{\ 2\mu}_{\lambda-\mu}\left(\frac{y^2 x^{2\mu}}{2}\right)$$

which is also known [3, p. 160(8·36)].

4. We add one more theorem before concluding the paper. This is also a general result from which many well-known results may be derived.

Theorem V.

If $$\phi(p,\ q) \underset{m,\ n\ ;\ \alpha,\ \beta}{\overset{H}{\rightleftharpoons}} f(x,\ y)$$

and

$$p^{-\alpha_1}\ q^{-\alpha_2}\ f\left(p^{\frac{-m_1}{m_2}}, q^{\frac{-n_1}{n_2}}\right) \underset{m,\ n\ ;\ \alpha,\ \beta}{\overset{H}{\rightleftharpoons}} g(x,\ y)$$

then

$$\phi(p,\ q) = \frac{\lambda\lambda'\ \mu\mu'\ m_1\ n_1}{m_2\ n_2} . c^{\frac{m_2}{m_1}\alpha_1} . d^{\frac{n_2}{n_1}\alpha_2} . c'^{\frac{m_1}{m_2}-1} . d'^{\frac{n_1}{n_2}-1} \times$$

$$\times\, p^{\frac{m_2}{m_1}\alpha_1+1} . q^{\frac{n_2}{n_1}\alpha_2+1} \int_0^{\infty}\int_0^{\infty} g(s,\ t)\, S^{\frac{m_1}{m_2}-1}\, t^{\frac{n_1}{n_2}-1}\, ds\, dt\ \times$$

$$\times\ H^{\,m+m',\,0}_{\,n+n',\,m+m'}\left[c\,p(c's)^{\frac{m_1}{m_2}}\middle|\begin{matrix}\left(a'_{n'}+c'-\frac{c'm_1}{m_2}, \frac{c'm_1}{m_2}\right), \left(a_n-\frac{c\,m_2\alpha_1}{m_1}, c\right)\\ \left(b_m-\frac{cm_2\alpha_1}{m_1}, c\right), \left(b'_{m'}+c'-\frac{c'\,m_1}{m_2}, \frac{c'm_1}{m_2}\right)\end{matrix}\right]$$

$$\times\ H^{\,\alpha+\alpha',0}_{\,\beta+\beta',\,\alpha+\alpha'}\left[dq(d't)^{\frac{n_1}{n_2}}\middle|\begin{matrix}\left(h'_{\beta'}+d'-\frac{d'n_1}{n_2}, \frac{d'n_1}{n_2}\right), \left(h_{\beta}-\frac{dn_2\alpha_2}{n_1}, d\right)\\ \left(f_{\alpha}-\frac{dn_2\alpha_2}{n_1}, d\right), \left(f'_{\alpha'}+d'-\frac{d'n_1}{n_2}, \frac{d'n_1}{n_2}\right)\end{matrix}\right]$$

provided the integrals involved converge absolutely.

Proof : Using the given relations we easily see that

$$\phi(p, q) = \lambda\lambda'\mu\mu' pq \int_0^\infty \int_0^\infty H_{n,m}^{m,o}\left[cpx \;\middle|\; \begin{matrix}(a_n, c)\\(b_m, c)\end{matrix} \right] H_{\beta,\alpha}^{\alpha,o}\left[dqy \;\middle|\; \begin{matrix}h_\beta, d)\\f_\alpha, d)\end{matrix} \right] \times$$

$$\times\, x^{\frac{-m_2}{m_1}(1+\alpha_1)} \cdot y^{\frac{-n_2}{n_1}(1+\alpha_2)}\, dx\, dy \times$$

$$\times \int_0^\infty \int_0^\infty H_{n',m'}^{m',o}\left[c' x^{\frac{-m_2}{m_1}} \cdot S \;\middle|\; \begin{matrix}(a'_{n'}, c')\\(b'_{m'}, c')\end{matrix} \right] H_{\beta',\alpha'}^{\alpha',0}\left[d' y^{\frac{-n_2}{n_1}} \cdot t \;\middle|\; \begin{matrix}(h'_{\beta'}, d')\\(f'_{\alpha'}, d')\end{matrix} \right] ds\, dt$$

This reduces to the result stated with the help of (1·7), (1·8) and (1·10).

Corollary : If $c = c' = d = d' = \frac{1}{2}$, $\lambda = \mu = \lambda' = \mu' = \frac{1}{2\sqrt{\pi}}$

$m = 2 = \alpha = m' = \alpha'$, $n = 0 = \beta = n' = \beta'$, $b_1 = \frac{1}{4} + \frac{\nu}{2}$,

$b_2 = \frac{1}{4} - \frac{\nu}{2}$, $f_1 = \frac{1}{4} + \frac{\mu}{2}$, $f_2 = \frac{1}{4} - \frac{\mu}{2}$, $b_1' = \frac{1}{4} + \frac{\nu_1}{2}$, $b'_2 = \frac{1}{4} - \frac{\nu_1}{2}$, $f'_1 = \frac{1}{4} + \frac{\mu_1}{2}$,

$f_2' = \frac{1}{4} - \frac{\mu_1}{2}$,

we arrive at the following :

$$\text{If} \qquad \phi(p, q) \xrightarrow[\nu,\ \mu]{K} f(x, y),$$

$$p^{-\alpha_1} q^{-\alpha_2} f\left(p^{\frac{-m_1}{m_2}}, q^{\frac{-n_1}{n_2}} \right) \xrightarrow[\nu_1,\ \mu_1]{K} g(x, y),$$

then,

$$\phi(p, q) = (2\pi)^{2-m_1-m_2-n_1-n_2} \cdot (2\, m_2)^{\frac{-m_2}{m_1}(\alpha_1+\frac{3}{2})+\frac{1}{2}} \times$$

$$\times\, (2n_2)^{\frac{-n_2}{n_1}(\alpha_2+\frac{3}{2})+\frac{1}{2})} \cdot p^{\frac{m_2}{m_1}(\alpha_1+\frac{3}{2})} \cdot q^{\frac{n_2}{n_1}(\alpha_2+\frac{3}{2})} \times$$

$$\times \int_0^\infty \int_0^\infty (st)^{\frac{1}{2}}\, g(s, t)\, ds\, dt \times$$

$$\times\, G_{0,2m_1+2m_2}^{2m_1+2m_2,0}\left[\left(\frac{p}{2m_2}\right)^{2m_2} \left(\frac{s}{2m_1}\right)^{2m_1} \middle|\, \triangle[m_2; \tfrac{1}{4}\{ 3 - \tfrac{m_2}{m_1}(2\alpha_1+3) \pm 2\nu\}] \triangle\left(m_1; \pm\frac{\nu_1}{2}\right) \right]$$

$$\times\, G_{0,2n_1+2n_2}^{2n_1+2n_2,\,0}\left[\left(\frac{q}{2n_2}\right)^{2n_2} \left(\frac{t}{2n_1}\right)^{2n_1} \middle|\, \triangle[n_2; \tfrac{1}{4}\{3 - \tfrac{n_2}{n_1}(2\alpha_2+3) \pm 2\mu\}],\ \triangle\left(n_1; +\frac{\mu_1}{2}\right) \right]$$

which is Rathie's Theorem [6, pp. 45, 46(15)].

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Bessel Transform

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1. Introduction

First R. V. Charchill (1954)[2], Tranter[11] established finite integral transforms with Kernel $P_n(x)$ and called them Legendre transforms. Scott[6] considered a more general case by choosing Jacobi polynomial $P_n^{(\alpha,\, \beta)}(x)$ for the Kernel. Sneddon[10] employed the finite Hankel transform for the solution of some initial and boundary value problems in the theory of elastic vibrations. Debnath[3], R. P. Singh[9] have studied integral transforms involving Laguerre and Hermite polynomials respectively.

Recently, following Tranter[11] and Scott[6], Lakshmanrao[5], Conte[1] and K. N. Shrivastava[7] have independently introduced a finite transform over the range $(-1, 1)$ with $(1-x^2)^{\nu-\frac{1}{2}} C_n^{\nu}(x)$ for its Kernel and called it Gegenbauer transform. This transform is defined as follows :

If $F(x)$ is a continuous function on the interval $-1 \leqslant x \leqslant 1$ its Gegenbauer transform is defined by the relation

$$f^{\nu}(n) = T\, F(x) = \int_{-1}^{+1} F(x)\,(1-x^2)^{\nu-\frac{1}{2}}\, C_n^{\nu}(x)\, dx, -\nu > -\tfrac{1}{2}, n = 0, 1, 2, \qquad (1)$$

where $C_n^{\nu}(x)$ is Gegenbauer polynomial.

The object of this paper is to introduce Bessel transform and to study its properties.

2. Definition and properties of Bessel transfrom

Bessel transform of $F(x)$ is denoted by $f(n)$ and is explained by the following relation, $f_k(n)$ has been shown as a transformation of $\frac{F(x)}{x^k}$ when $K=0, 1, 2, \ldots\ldots m.$

$$f(n) = T\{F(x)\} = \frac{1}{2\pi i}\int_c e^{-2/x}\, Y_n(x)\, F(x)\, dx,\ 0 \leqslant x \leqslant 2\pi,\ n = 0, 1, 2, \qquad (2)$$

where $F(x)$ is analytic function within and on the boundary of a unit circle $Y_n(x)$ being Bessel polynomial of degree n. The Bessel $Y_n(x)$ is solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + (2x+2)\frac{dy}{dx} - n(n+1)\,y = 0 \qquad (3)$$

where $n = 1, 2, 3, \ldots\ldots$

which is of the form in standard Hypergeometric notation as

$$Y_n(x) = \sum_{k=0}^{n} \frac{1}{k!} \frac{(n+k)!}{(n-k)!} (x/2)^k = {}_2F_0[-n, 1+n; -; -\tfrac{1}{2}x]$$

Now the self adjoint differential form of $Y_n(x)$ is

$$\frac{d}{dx}\left[x^2 e^{-2/x} \frac{d}{dx} F(x) \right] = n(n+1) e^{-2/x} F(x) \tag{4}$$

The orthogonal property for Bessel polynomial as proved by Krall and Frink[4] is

$$\frac{1}{2\pi i} \int_c e^{-2/x} Y_n(x) Y_m(x)\, dx = \begin{cases} 0 & (n \neq m) \quad (5) \\ \delta m\, n & (n = m) \quad (6) \end{cases}$$

where $\delta m\, n = (-1)^{n+1} 2/(2n+1)$

The most fundamental property of Bessel transforms is contained in the following theorem.

3. Theorem : 1. Under transformation T differential form :

$$\lambda[F(x)] = x^2 F''(x) + (2x + 2) F'(x)$$

is reduced to algebraic form

$$T[\lambda \{F(x)\}] = n(n+1) f(n)$$

Proof : Suppose that $[\lambda \{ F(x) \}]$ denote differential fo m

$$\lambda[F(x)] = e^{2/x} \frac{d}{dx}\left[x^2 e^{-2/x} \frac{d}{dx} F(x) \right] \tag{7}$$

then

$$T[\lambda \{F(x)\}] = \frac{1}{2\pi i} \int e^{-2/x} Y_n(x) \lambda[F(x)]\, dx \tag{8}$$

Substituting the value of $\lambda[F(x)]$ in (8) from (7), we obtain

$$T[\lambda \{F(x)\}] = \frac{1}{2\pi i} \int_c e^{-2/x} Y_n(x) \left[e^{2/x} \frac{d}{dx} \left\{ x^2 e^{-2/x} \frac{d}{dx} F(x) \right\} \right] dx$$

$$= \frac{1}{2\pi i} \int_c Y_n(x) \frac{d}{dx} \left[x^2 e^{-2/x} \frac{d}{dx} F(x) \right] dx$$

$$= \frac{1}{2\pi i} \int_c e^{-2/x} Y_n(x)\, n(n+1) F(x)\, dx$$

$$= n(n+1) \frac{1}{2\pi i} \int_c e^{-2/x} Y_n(x) F(x)\, dx$$

$$= n(n+1) f(n)$$

4. Inverse Transformation : Under appropriate conditions a function $F(x)$ may be expanded in a series of Bessel Polynomials

$$F(x) = \sum_{n=0}^{\infty} c_n Y_n(x)$$

where $$c_n = (\delta\, mn)^{-1} \frac{1}{2\pi i} \int_c e^{-2/x}\, Y_n(x)\, F(x)\, dx$$

$$= (\delta\, mn)^{-1} f(n)$$

The inversion which gives the function $F(x)$ in the terms of its transform is

$$F(x) = \sum_{n=0}^{\infty} (\delta\, mn)^{-1} f(n)\, Y_n(x)$$

$$= T^{-1}\{f(n)\} \quad 0 \leqslant x \leqslant 2\pi$$

5. Theorem 2. If $F(x)$ be analytic function within and on the boundary of a unit circle, and $T[F(x)]$ exists then $T[\lambda\{F(x)\}]$ also exists

and $$T[\lambda\{F(x)\}] = n(n+1) f(n) \tag{9}$$

If $\lambda[F(x)]$ and $F(x)$ satisfy the condition of theorem 1 then

$$T[\lambda^2\{F(x)\}] = T[\lambda\{\lambda[F(x)\}]$$

$$= n^2(n+1)^2 f(n)$$

Similarly $$T[\lambda^3\{F(x)\}] = n^3(n+1)^3 f(n)$$

and so on up to m times. If $\lambda^m[F(x)]$ and $F(x)$ satisfy the condition of theorem 1

$$T[\lambda^m\{F(x)\}] = n^m (n+1)^m f(n)$$

where $$m = 1, 2, 3, \ldots\ldots k$$

If $T[F_1(x)]$ and $T[F_2(x)]$ both exists,

then

$$T[C_1 F_1(x) + C_2 F_2(x)] = C_1\, T[F_1(x)] + B_2\, T(F_2(x)]$$

where C_1 and C_2 are constants and transformation of $T[F_1(x)]$ and $T[F_2(x)]$ are linear and satisfy the condition of theorem 1. Assuming further that $F(x)$ is even, we have

$$T[F(-x)] = \frac{1}{2\pi i} \int_c e^{-2/x}\, Y_n(x)\, F(-x)\, dx$$

$$= f(n)$$

6. Examples :

I. let $F(x)$ be any polynominal of decree m then

$$f(n) = 0 \text{ if } n > m$$

II. let $F(x) = Y_n(x)$

$$T\{Y_m(x)\} = \frac{1}{2\pi i} \int_c e^{-2/x}\, Y_n(x)\, Y_m(x)\, dx \quad \begin{cases} 0 & (n \neq m) \\ \delta_{mn} & (n = m) \end{cases}$$

III. let $F(x) = x^k$.

then

$$T\{F(x)\} = T\{x^k\}$$

$$= \frac{1}{2\pi i}\int_c e^{-2/x}\, Y_n(x)\, x^k\, dx$$

$$= \frac{(-2)^{k+1}\, k^{(n)}}{(n+k+1)!}$$

where $k^{(n)} = k(k-1)(k-2)\ldots.(k-n+1)$

IV. We have the relation (W. Al. Salam)

$$Y_n^{a}(x) = n!\left(-\frac{x}{2}\right)^n L_n^{(-2n-a-n+1)}\left(\frac{2}{x}\right)$$

for $a = 0$ we have

$$Y_n(x) = n!\left(-\frac{x}{2}\right)^n L_n^{(-2n-1)}\left(\frac{2}{x}\right)$$

Therefore $$T\left\{n!\left(-\frac{x}{2}\right)^n L_n^{(-2n-1)}\left(\frac{2}{x}\right)\right\}$$

$$= \frac{1}{2\pi i}\int_c e^{2/x}\, Y_n(x)\left\{n!\left(-\frac{x}{2}\right)^n L_n^{(3n-1)}\left(\frac{2}{x}\right)\right\} dx$$

$$= \frac{1}{n!}\,\delta mn$$

V. We have

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n}\binom{n+\alpha}{n-m}\left(\frac{-x}{m!}\right)^m$$

then

$$T\{L_n^{(\alpha)}(x)\} = \frac{1}{2\pi i}\int_c e^{-2/x}\, Y_n(x)\, L_n^{(\alpha)}(x)\, dx$$

$$= \sum_{m=0}^{n}\frac{(-2)^{k+1}}{(n+k+1)!}\,\frac{(-1)^m}{m!}\binom{n+\alpha}{n-m}$$

7. Transfer of the Derivative of of F(x) :

Let $F'(x)$ be analytic and function of x

Then

$$T\{F'(x)\} = \frac{1}{2\pi i}\int_c e^{-2/x}\, Y_n(x)\, F'(x)\, dx$$

$$= -\frac{1}{2\pi i}\int_c \frac{d}{dx}\left[e^{-2/x}\, Y_n(x)\right] F(x)\, dx$$

$$= -\frac{1}{2\pi i}\int_c \frac{2}{x^2} e^{-2/x} Y_n(x) F(x)\, dx$$

$$-\frac{n}{2\pi i}\int_c \frac{1}{x} e^{-2/x} Y_n(x) F(x)\, dx -$$

$$\frac{1}{2\pi i}\int_c \frac{1}{x^2} e^{-2/x} Y_{n-1}(x) F(x)\, dx$$

(on integration by parts)

$$\therefore\ T\{F'(x)\} = -2 f_2(n) - n f_1(n) - f_2(n-1)$$

$$T\{F''(x)\} = \frac{1}{2\pi i}\int_c e^{-2/x} Y_n(x) F''(x)\, dx$$

on integration by parts

$$T(F''(x)\} = -\frac{1}{2\pi i}\int_c \frac{d}{dx}[e^{-2/x} Y_n(x)] F'(x)\, dx$$

$$= -\frac{1}{2\pi i}\int_c \frac{2}{x^2} e^{-2/x} Y_n(x) F'(x)\, dx$$

$$-\frac{n}{2\pi i}\int_c \frac{e^{-2/x}}{x} Y_n(x) F'(x)\, dx - \frac{1}{2\pi i}\int_c \frac{1}{x^2} e^{-2/x} Y_{n-1}(x) F'(x)\, dx$$

$$= \frac{2}{x^2}\ T\{F'(x)\} - \frac{1}{2\pi i}\int_c \frac{2}{x^3} e^{-2/x} Y_n(x) F(x)\, dx$$

$$+\frac{n}{x}\ T\{F'(x)\} - \frac{n}{2\pi i}\int_c \frac{1}{x^2} e^{-2/x} Y_n(x) F(x)\, dx$$

$$+\frac{1}{2\pi i}\int_c \frac{2}{x^4} e^{-2/x} Y_{n-1}(x) F(x)\, dx - \frac{1}{2\pi i}\int_c \frac{2}{x^3}$$

$$\times e^{-2/x} Y_{n-1}(x) F(x)\, dx + \frac{1}{2\pi i}\int_c \frac{1}{x^4} e^{-2/x} Y_n(x)$$

$$\times F(x)\, dx - \frac{n}{2\pi i}\int_c \frac{1}{x^3} e^{-2/x} Y_{n-1}(x) F(x) dx$$

$$-\frac{1}{2\pi i}\int_c \frac{1}{x^2} e^{-2/x} Y_{n-1}(x) F(x) dx$$

Therefore

$$T\{F''x)\} = 3 f_4(n) + (3n-2) f_3(n) + 4 f_4(n-1) - 2 f_3(n-1) - f_2(n-1)$$

and so on for higher derivatives.

8. In conclusion :

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Integrals involving Bessel coefficients of two arguments

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Abstract

A few integrals involving Bessel coefficients of two arguments are evaluated.

1. Introduction

The two types of the Bessel coefficients $J_m^{(1)}(x, y)$ and $J_m^{(2)}(x, y)$ respectively are defined by

$$e^{\frac{1}{2}x\left(u-\frac{1}{u}\right)+\frac{1}{2}y\left(u^2-\frac{1}{u^2}\right)} = \sum_{m=-\infty}^{\infty} u^m J_m^{(1)}(x, y), \tag{1}$$

$$e^{\frac{1}{2}x\left(u-\frac{1}{u}\right)+\frac{1}{2}y\left(u^2+\frac{1}{u^2}\right)} = \sum_{m=-\infty}^{\infty} u^m J_m^{(2)}(x, y), \tag{2}$$

for all finite values of the arguments x, y and the parameter u provided $u \neq 0$.

It is found that

$$J_m^{(1)}(x, y) = \frac{1}{\pi}\int_0^{\pi} \cos(m\theta - x\sin\theta - y\sin 2\theta)\, d\theta, \tag{3}$$

$$J_m^{(2)}(x, y) = \frac{1}{\pi}\int_0^{\pi} e^{y\sin 2\theta}\cos(m\theta - x\sin\theta)\, d\theta, \tag{4}$$

where $\qquad m = 0, \pm 1, \pm 2, \ldots .$

These coefficients are the particular cases of the generalised Bessel coefficients defined in[2].

2. Integrals

If we take

$$\psi = \phi - x\sin\phi - y\sin 2\phi, \tag{5}$$

then the Fourier's cosine series expansion gives that

$$\frac{1}{1 - x\cos\phi - 2y\cos 2\phi} = 1 + \sum_{m=1}^{\infty} J_m^{(1)}(mx, my)\cos m\psi. \tag{6}$$

Integrating this with respect to ϕ from 0 to π, we get

$$\frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{1 - x\cos\phi - 2y\cos 2\phi} = 1 + \sum_{m=1}^{\infty}\left\{J_m^{(1)}(mx, my)\right\}^2. \tag{7}$$

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On evaluating the left hand side integral in this, we get

$$1+2\sum_{m=1}^{\infty}\{J_m^{(1)}(mx, my)\}^2 = \frac{1}{\sqrt{x^2+16y(1+2y)}}\left[\frac{\alpha_1}{\sqrt{1-\alpha_1^2}} - \frac{\alpha_2}{\sqrt{1-\alpha_2^2}}\right] \tag{8}$$

where α_1, α_2 are the roots of the equation

$$(1+2y)\,\alpha^2 - x\alpha - 4y = 0$$

such that α_1 corresponds to $\alpha = x$ as $y \to 0$ and

$$x^2+8y^2 \pm x\sqrt{x^2+16y(1+2y)} < 2.$$

In particular, if we put $y = 0$, we get [1, p. 124].

$$\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{1-x\cos\phi} = 1+2\sum_{m=1}^{\infty} J_m^{\,2}(mx) = \frac{1}{\sqrt{1-x^2}},\ x^2 < 1. \tag{9}$$

Now we make justifiable integration of both the sides of the result

$$\int_0^{\infty} e^{-ax+cx\cos 2\theta}\cos(bx\sin\theta)\,d\theta = \frac{a-c\cos 2\theta}{(a-c\cos 2\theta)^2+b^2\sin^2\theta},\ a \pm c > 0$$

with respect to θ from 0 to π, we get

$$\int_0^{\infty} e^{-ax} J_0^{(2)}(bx, cx)\,dx = \frac{1}{\pi}\int_0^{\pi}\frac{a-c\cos 2\theta}{(a-c\cos 2\theta)^2+b^2\sin^2\theta}\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\frac{2c\sin^2\theta+(a-c)}{4c^2\sin^4\theta+\sin^2\theta\,(4c(a-c)+b^2)+(a-c)^2}\,d\theta$$

$$= \frac{2}{(1+\alpha)\pi}\int_0^{\pi/2}\left[\frac{1}{2c/\alpha\,\sin^2\theta+(a-c)}+\frac{\alpha}{2c\alpha\sin^2\theta+(a-c)}\right]d\theta$$

$$= \frac{1}{(1+\alpha)\sqrt{a-c}}\left[\frac{1}{\sqrt{(a-c)+2c/\alpha}}+\frac{\alpha}{\sqrt{(a-c)+2c\alpha}}\right], \tag{10}$$

where $\alpha = \dfrac{4c(a-c)+b^2+b\sqrt{b^2+8c(a-c)}}{4c\,(a-c)}$ and $a - c > 0$.

On putting $c = 0$, we get the known result [1, p. 57]

$$\int_0^{\infty} e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}},\ \text{for } a > 0.$$

Similarly, if we consider the result

$$\int_0^{\infty} e^{-ax+ax\cos 2\theta}\cos(bx\sin\theta)\,dx = \frac{2a}{4a^2\sin^2\theta+b^2},\ a > 0$$

we get

$$\int_0^{\infty} e^{-ax} J_0^{(2)}(bx, ax)\,dx = \frac{2a}{b\sqrt{4a^2+b^2}},\ a > 0, \tag{11}$$

For further deductions we may follow [1, p. 58].

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Nitrogen Transformations in Soil—Effect of nitrogenous fertilizer, organic matter and phosphate

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Nitrogen being the main limiting factor in crop production increasing quantities of nitrogenous fertilizers are being used for stepping up crop yields. On the application of artificial fertilizers to the soil a good deal of chemical nitrogen is lost through leaching, denitrification and volatilisation. On the basis of field experiments conducted on the broad-balk wheat fields at Rothamsted Russell (1950) reported loss of fertilizer nitrogen to the extent of 50 per cent. Mac Vicar *et al.* (1951) using tracer technique reported 38–47% recovery of the inorganic nitrogen by oats and sudan grass crops. According to Bartholomew and Hiltbold (1952) the present recovery of tagged nitrogen varied from 27 to 57% with two rates of its application. Evolving suitable methods for reducing the huge loss of nitrogen so as to have maximum benefit from the costly fertilizers warrants investigation.

Harmsen and Vanschreven (1955) and Allison (1955) made exhaustive review of the pertinent literature and emphasized use of straw and sawdust etc. for preventing nitrogen losses from soil. Chandra and Bollen (1959) on the basis of green house studies suggested use of wheat straw and sawdust as a source of organic matter for conservation of nitrogen. In India straw being used for cattle feeding, a cheap organic matter source—sugarcane trash has been tried which can easily be diverted from fuel to the fields. In view of the nitrogen conserving action of the phosphates (Crowther and Yates, 1941, Russell, 1950 and Dhar, 1961) the effect of Tata basic slag—a cheap source of phosphate has also been investigated.

Materials and Methods

The collected soil was passed through 100 mesh sieve after drying in air. Its chemical composition was determined by the methods described by Piper (1950) and A. O. A. C. (1955). Sugarcane trash and Tata basic slag were finely ground so as to pass through 10 mesh and 100 mesh sieve respectively. Two Kilograms of soil were filled in each of the required number of pots and the treatments shown in table–2 were set up. Organic material, ammonium sulphate and slag were used at the rate of 2000 ppm carbon, 100 ppm nitrogen and 50 ppm P_2O_5 respectively. Throughout the experiment the moisture content was maintained at 50 percent of water holding capacity by adding distilled water daily. The mean average temperature during the course of the experiment was 31°C. After definite intervals of time soil samples were drawn and analysed. The total nitrogen was determined by the Kjeldahl salicylic acid reduction method (John Brooks, 1936) to include nitrate nitrogen. Ammoniacal nitrogen was determined by the method of Shrikhande (1943) and nitrate nitrogen by Phenol disulphonic

acid method. The pH was determined by glass electrode in soil : water suspension 1 : 2. Total bacterial and Azotobacter number were determined by the plate dilution technique (Allen, 1957) using Thornton and Martin's media.

Results and Discussion

TABLE 1

Composition of the soil and amendments

Analysis of Soil

Loss on ignition	2·96	*Mechanical analysis*	
Nitrogen :—		Sand	64·9%
Ammonium	0·0012%	Silt	14·5%
Nitrate	0·0017%	Clay	19·9%
Total	0·0348%		
Total Carbon	0·253%		
C/N	7·2	*HCl extract analysis*	
pH	7·6	HCl. insoluble	81·2%
C. E. C.	12·6 me/100 gm soil	R_2O_3	11·05%
Exch. Ca.	5·1 me/100 gm soil	Fe_2O_3	4·98%
Available P_2O_5	14 ppm	CaO	1·42%
Total bacteria	8·6 millions/gm	MgO	1·01%
Azotobacter number	1·0 millions/gm	K_2O	1·04%
Fungus count	40,000/gm	P_2O_5	0·11%

Analysis of sugarcane trash and Tata basic slag

Sugarcane Trash		*Tata basic slag*	
Total Carbon	40·9%	Total P_2O_5	7·91%
Total Nitrogen	0·21%	Avail. P_2O_5	3·9%
C/N	194·7	CaO	30·37%
		K_2O	0·98%

The results recorded in table-2 show that when ammonium sulphate was added to the soil, in first fifteen days 55 percent and 61 percent of it were nitrified in presence and absence of slag respectively. Afterwards nitrification percentage increased to 77 and 82 in 45 days but again dropped to 63 and 72 by the end of 75 days. This shows that the nitrification is faster in the beginning and after reaching its maximum registers decline. A drop in nitrification observed after 45 days appears to be due to domination of denitrification over nitrification as also revealed by the total nitrogen values. This depression in nitrification relative to soil corroborates the findings of Waksman (1927), Allison and Anderson (1951) and Chandra and Bollen (1959) where as the beneficial action of phosphates on nitrification is in line with the observations of Jaiswal (1964).

By the addition of sugarcane trash nitrification dropped to 35 percent and 37 percent in 45 days, which however, rose to 53 percent and 62 percent by the end of 75 days. This temporary depression in nitrification seems to be due to assimilation of nitrate nitrogen by rapidly multiplying micro-organisms on the

TABLE 2

Effect of sugarcane trash, ammonium sulphate and Tata basic slag on nitrogen transformations

Treatments	Ammoniacal Nitrogen ppm Period in days.				Nitrate Nitrogen ppm Period in days.				Total Nitrogen ppm Period in days.				pH Period in days.				Bacterial count millions/gm. Total bacteria On 45th day.	Azoto-bacter
	0	15	45	75	0	15	45	75	0	15	45	75	0	15	45	75		
Soil alone	13	20	26	20	17	21	33	28	350	368	380	368	7·1	6·9	7·0	6·8	10·7	1·6
Soil + amm. sulphate.	109	58	44	41	16	76	110	91	446	420	419	412	6·9	6·8	6·5	6·5	11·5	1·0
Soil + amm. sulphate + slag	109	57	51	46	17	82	115	100	443	422	428	425	7·2	7·1	6·9	6·9	13·5	1·2
Soil + trash.	12	15	29	40	15	8	12	27	354	384	398	398	7·1	6·9	6·8	6·6	20·0	3·4
Soil + trash + slag.	11	11	36	48	17	9	12	33	347	394	410	412	7·1	6·8	6·9	6·8	28·5	5·0
Soil + trash + amm. sulphate.	108	74	63	58	16	42	68	81	452	462	417	476	7·0	6·8	6·6	6·5	38·5	4·5
Soil + trash + amm. sulphate + slag.	111	77	70	58	18	40	70	90	449	466	484	490	7·2	7·0	6·8	6·7	40·0	4·6

added energy source as reflected by the total bacterial numbers. Subsequent increase in nitrification was as a result of mineralisation of immobilized nitrogen. The increase in total nitrogen values by addition of trash shows that it not only conserves added nitrogen but also favours nitrogen fixation. These results are in agreement with those of Chandra and Bollen (1959) who recorded nitrogen fixation even in presence of nitrogenous fertilizer under laboratory conditions with the addition of straw and saw dust.

An increase in total nitrogen values with the addition of trash may be due to suppression of denitrification and fixation of atmospheric nitrogen by biological and physico-chemical agencies, the oxidation of organic matter providing energy for the endothermic process (Mc Garity *et al*, 1958, Chandra *et al*, 1957 and Dhar, 1961). Bremmer and Shaw (1958) while studying the effect of energy material on reduction of nitrates observed that the maximum loss occurred with glucose and nitrate at C : N ratio of 5 : 1, with straw and nitrate at a C : N ratio of 30 : 1 and with wider rations the loss of nitrogen decreased due to nitrogen fixation. Addition of trash with very wide C : N ratio and with the resulting widening of C : N ratio of the mixture might have been helpful in nitrogen fixation.

It is very fascinating to record that in presence of basic slag both the nitrogen fixation and mineralisation were enhanced. Increased nitrogen fixation in presence of phosphates is in agreement with the findings of Dhar *et al.* (1961). In all nitrogenous transformations nitrite ions are formed from the oxidation of ammonia and its salts as well as by the reduction of nitrates, Dhar pointed out 40 years ago that the unstable and explosive substance ammonium nitrite is always formed when organic or inorganic nitrogenous compounds are applied in crop production. Ammonium nitrite breaks with large evolution of heat into nitrogen gas and water as in the equation :

$$NH_4\,NO_2 = N_2 + 2H_2O + 718\text{ K Cal.}$$

In 1920 Dhar reported that this reaction is catalysed by acids. Basic slag being alkaline in nature reduces the decomposition of ammonium nitrite by decreasing acidity and supplying calcium, magnesium and potassium ions to form calcium, magnesium and potassium nitrites which are more stable than ammonium nitrite and thus increase the concentration of available nitrogen. Dhar has proved that the low recovery of nitrogen in crop yield is caused by the formation and decomposition of the unstable substance ammonium nitrite.

These results show that organic materials of wide C : N ratio like trash especially with slags could profitably be used for conservation of soil and fertilizer nitrogen and thus could add to efficiency of these costly fertilizers.

Summary

The effect of sugarcane trash, ammonium sulphate and tata basic slag on nitrogen transformations in soil were studied. Sugarcane trash added with ammonium sulphate not only conserved fertilizer nitrogen but was also helpful in nitrogen fixation. Temporary exhaustion of available nitrogen by rapidly multiplying microbes showed that in field practice these energy material with such wide C : N ratio should be applied several weeks before the expected nitrogen need of the plants. Tata basic slag also helped in conservation and mineralization of nitrogen. Thus organic materials like sugarcane trash and phosphates could be used not only for increasing humus reserves of the soil thus creating favourable conditions for plant growth but also for conserving fertilizer nitrogen.

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Non Radial Hydromagnetic Oscillations of an incompressible cylinder

By

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Abstract

The present paper deals with the small oscillations of an incompressible homogeneous cylinder of an infinite length and zero resistivity under its own gravitation in presence of a magnetic field having both toroidal and poloidal components which are functions of r only. The differential equation for $r\xi_r$ has been solved by the method of series solution. The cases of $m = o$, and $m = 1$ are to be disregarded as these give radial displacement on the axis. The case $k = o$ has been separately dealt with. The frequency and amplitude of oscillation have also been calculated for $m = 2$.

1. Introduction

The interest in the stablity of cylindrical systems and in the computation of proper frequencies of oscillations has been revived in recent years by certain astronomical and physical problems : the stability of spiral arms of galaxies and the stability of the so called pinch configurations.

The problem of small pulsations of a homogeneous cylinder in presence of magnetic field parallel to its axis was considered by Chandrasekhar and Fermi[1], and by Lyttkens[2], in the particular case of radial oscillations of a compressible perfect gas. Bhatnagar and Nagpaul[3] considered the problem to study the effect of finite conductivity. The problem was reconsidered by Simon[4] with a uniform magnetic field parallel to the axis of the cylinder throughout the space. He proved that the vacuum magnetic field has a stabilizing effect over that of inside magnetic field. We have studied the same problem with a variable magnetic field having both toroidal and poloidal components depending upon r only. This is to add to our[9] problem studied for variable axial magnetic field only. The small oscillations of an incompressible homogeneous cylinder of an infinite length and zero resistivity under its own gravitiational field have been studied. The displacement currents have been neglected. It is supposed that the magnetic field in the interior and the exterior has the following components.

(1) $$H_r = 0, H\phi = Kr,$$

(2) $$\text{and } H_z = H_c\left(1 - a\frac{r^2}{R^2}\right),$$

(3) $$(H_r)_e = 0, (H\phi)_e = \frac{KR^2}{r},$$

(4) $$(H_z)_e = H_c(1 - a),$$

Where subscript e refers to the exterior of the Cylinder. Here the axial field is square of the field taken previously[9]. The toroidal components as mentioned in equation (1) and (3) are additional features of the problem.

2. Equilibrium

Consider a material cylinder, homogeneous and incompressible of radius R, and infinite length, in equilibrium under its own gravitation. The cylinder is surrounded by an infinite vacuum. From the hydrostatic equation of equilibrium we have

(5) $$\frac{dp}{dr} + \rho \frac{dV_i}{dr} - \frac{1}{4\pi}\left[(\text{curl } \overline{H}_i) \times \overline{H}_i \right] = 0.$$

The equilibrum magnetic field is given by

(6) $$\overline{H}_i = (0, H_\phi, H_z),$$

and

(7) $$\overline{H}_e = [0, (H_\phi)_e, (H_z)_e],$$

with the components as given in equations (1) to (4).

From the equations of Laplace and Poisson we have

(8) $$V_e = 2\pi G \rho R^2 \log \frac{r}{R} + c_0 ,$$

(9) $$V_i = \pi G \rho r^2,$$

Also integration of equation (5) gives

(10) $$p = \pi G\rho R^2 \left(1 - \frac{r^2}{R^2}\right) - \frac{2 K^2 r^2 + H_c^2 \left(1 - \frac{ar^2}{R^2}\right) - 2K^2 R^2}{8\pi} + \frac{H_c^2 (1-a)^2}{8\pi},$$

grad V and $\left[p + \frac{1 H 1^2}{8\pi} \right]$ are continuous on the boundary of the cylinder.

3. The equations of motion

We consider the small oscillations so that we neglect the second and higher order quantities.

Let $\overline{\xi}$ denotes the Eulerian displacement with respect to the previous equilibrium. We have the following equations governing the material motion.

(11) $$\text{div } \overline{\xi} = 0 ,$$

(12) $$\rho \frac{d^2 \overline{\xi}}{dt^2} = - \text{grad } \Delta p - \rho \text{ grad } \Delta V_i + \frac{1}{4\pi} [(\text{curl } \overline{H}_i) \times \Delta \overline{H}_i + (\text{curl } \Delta \overline{H}_i) \times \overline{H}_i] ,$$

(13) $$\Delta \overline{H}_i = \text{curl } (\overline{\xi} \times \overline{H}_i) ,$$

(14) $$\nabla^2 \Delta V_i = 0,$$

where Δ denotes Eulerian variation.

The corresponding equations in the vacuum are

(15) $$\nabla^2 \triangle V_e = 0$$

(16) $$\text{curl } \bar{H}_e = 0, \text{ div } H_e = 0.$$

These admit solutions of the only type

(17) $$(H\phi)_e = \frac{C}{r}, (H_z)_e = D, \text{ where } C \text{ and } D \text{ are constants.}$$

Equations (17) in the absence of surface currents give

(18) $$C = KR^2, \ D = H_0(1 - a).$$

The assumed form of the vacuum magnetic field agrees with the solution of equations (16).

Also the perturbation in the magnetic field in the vacuum, neglecting the displacement currents is governed by

(19) $$\text{div } \triangle \bar{H}_e = 0, \text{ curl } \triangle \bar{H}_e = 0,$$

and the electric field in the vacuum is given by

(20) $$\text{curl } \bar{E}_e = -\frac{1}{c}\frac{\partial}{\partial t}(\triangle \bar{H}_e).$$

4. The differential equation

Consider first, the quations valid inside the cylinder.

(21) $$\frac{d^2\bar{\xi}}{dt^2} = -\text{grad } \zeta + \frac{1}{4\pi\rho}[(\text{curl } \bar{H}_i)\times \triangle \bar{H}_i + (\text{curl } \triangle \bar{H}_i)\times \bar{H}_i],$$

where

(22) $$\zeta = \frac{\triangle p}{\rho} + \triangle V_i.$$

Equation (11), (13), (14) and (21) completely determine all the physical quantities $\bar{\xi}$, ζ, and $\triangle Vi$ involved in the oscillations of the cylinder. Equations (15) and (19) determine the oscillations of the vacuum gravitational and magnetic fields.

We consider normal modes of oscillations in which each physical quantity varies in the Eulerian mode of pertu bations as

(23) $$f(r, \phi, z, t) = f(r) \exp. [\, i\,(\sigma\, t + m\,\phi + kz)\,].$$

where σ is the circular frequency of oscillations, m, the azimuthal number, a positive or negative integer and k a real constant which denotes the wave number.

From equations (11), (13) and (21), we have

(24) $$\sigma^2 \xi_r = \frac{1}{4\pi\rho}\frac{\partial}{\partial r}\Big[\, H_z^2\, ik\xi_z + KH_z\, im\, \xi_z + KH_z\, ikr\, \xi\phi +$$

$$+ K^2\, im\, r\, \xi_\phi \Big] + \frac{KH_z}{4\pi\rho}\Big[2mk\, \xi_r + 2ik\, \xi_\phi + \cdot\ \frac{K^2}{4\pi\rho}\Big[2\, im\, \xi_\phi + m^2 \xi_r \Big]$$

$$+ \frac{H_z{}^2\, k^2\, \xi_r}{4\pi\delta} - \frac{1}{4\pi\rho}\, \frac{\partial}{\partial r}\Big[\xi_r\, H_z\, \frac{\partial\, H_z}{\partial r} \Big],$$

(25) $$\sigma^2\, \xi_\phi = \frac{im}{r}\, \zeta + \frac{H_z{}^2}{4\pi\rho}\Big[k^2\, \xi_\phi - \frac{mk}{r}\, \xi_z \Big] - \frac{2k^2}{4\pi\rho}\, im\, \xi_r +$$

$$- \frac{1}{4\pi\rho}\, im\, \xi_r\, H_z \frac{\partial\, H_z}{\partial r} - \frac{KH_z}{4\pi\rho}\Big[\frac{m^2}{r}\, \xi_z - mk\, \xi_\phi + 2\, ik\, \xi_r \Big],$$

(26) $$\sigma^2\, \xi_z = ik\, \zeta - \frac{H_z}{4\pi\rho}\, \frac{\partial\, H_z}{\partial r}\, ik\, \xi_r - \frac{KH_z}{4\pi\rho}\Big[k^2\, r\xi_\phi - mk\, \xi_z \Big]$$

$$+ \frac{K^2}{4\pi\rho}\, [m^2 \xi_z - mkr\, \xi_\phi],$$

(27) $$(r\, \xi_r)' + im\, \xi_\phi + ikr\, \xi_z = 0,$$

with

(28) $$\frac{\partial}{\partial t} = i\sigma\,,\ \frac{\partial}{\partial \phi} = im\,,\ \frac{\partial}{\partial z} = ik$$

Case I: $k \neq 0$

Eliminating ξ_ϕ from equations (24–25) with the help of equation (26) and solving for ξ_z and ζ in terms of $(r\xi_r)$ we find

(29) $$\xi_z \zeta = \frac{Pikr\, (r\, \xi_r)' + 2ik\, \Big[\frac{KH_z\, mk}{4\pi\rho} + C^2\, m^2 \Big] (\, r\, \xi_r\,)}{(m^2 + k^2 r^2)\, P}$$

(30) $$\zeta = \frac{1}{(m^2 + k^2\, r^2) P}\Big[\, r(r\, \xi_r)'\, \sigma^2\, P + r\, \xi_r\, \Big\{ -P(m^2 + k^2 r^2)\, \frac{2A^2.\, a}{R^2}$$

$$+ 2\, \Big(\frac{KH_z mk}{4\pi\rho} + C^2\, m^2 \Big)\Big(D - \frac{KH_z mk}{4\pi\rho} \Big) - \frac{k^2}{m^2}\, r^2 \Big(\frac{KH_z mk}{4\pi\rho} + c^2 m^2 \Big)\Big\}\Big]$$

(31) $$P = D - \frac{2mk\, KH_c}{4\pi\rho}\Big(1 - \frac{ar^2}{R^2} \Big) - \frac{H_c{}^2 k^2}{4\pi\rho}\, \Big(1 - \frac{ar^2}{R^2} \Big)^2,$$

and

(32) $$D = \sigma^2 - m^2\, C^2,\ C^2 = \frac{K^2}{4\pi\rho}.$$

In all the equations dashes denote differentiation with respect to r.

Equation (24) now with the help of equations (26) and (30) reduces to

(33) $$\text{Pik}\, \xi_r = - \frac{2\, ik}{m^2}\, (r\, \xi_r)'\, \Big[\frac{KH_z mk}{4\pi\rho} + 2C^2\, m^2 \Big]$$

$$+ \frac{\partial}{\partial r}\, (P\xi_z) + \frac{2k^2}{m^2}\Big[\frac{KH_z mk}{4\pi\rho} + C^2\, m^2 \Big]\, r\xi_z.$$

Substituting for ξ_z from (29), we at once write down the required differential equation of $(r\,\xi_r)$ with regular singularity at the axis of the the cylinder $r = 0$ as

(34) $$r^2(r\,\xi_r)''\,[P^2(m^2 + k^2r^2)] + r\,(r\,\xi_r)'\left[m^2\left(P^2 + r\,P\frac{\partial P}{\partial r}\right) + \right.$$

$$\left. + k^2r^2\left(-P^2 + r\,P\frac{\partial P}{\partial r}\right)\right] + (r\,\xi_r)\left[-m^4P^2 + k^2r^2\right.$$

$$\left\{4\left(\frac{KH_zmk}{4\pi\rho} + C^2m^2\right)\left(-P + \frac{KH_zmk}{4\pi\rho} + C^2m^2\right) - 2m^2P^2 - \frac{4KH_cm^2}{k}\,\frac{Pa}{R^2}\right\} +$$

$$\left. + k^4r^4\left\{\frac{4}{m^2}\left(\frac{KH_zmk}{4\pi\rho} + C^2m^2\right)^2 - P^2 - \frac{4\,KH_cm^2}{k}\cdot\frac{Pa}{R^2}\right\}\right].$$

On substituting for P' from (31) dividing by D^2 throughout ($D \neq 0$) and making the following substitutions.

(35) $$(r\,\xi_r) = u\;;\; t = \frac{r^2}{R^2}\;;\; \lambda = kR,$$

(36) $$b^2 = \frac{H_c{}^2\,k^2}{4\pi\rho\,D}\;;\; c^2 = \frac{C^2}{D},$$

the differential equation (34) comes out to be of the form

(37) $$4t^2\frac{d^2u}{dt^2}\,[A_0m^2 + A_1t + A_2t^2 + A_3t^3 + A_4t^4 + A_5t^5] +$$

$$+ 4t\frac{du}{dt}\,[A_0m^2 + B_1t + B_2t^2 + B_3t^3 + B_4t^4 + B_5t^5]$$

$$+ u\,[-m^4A_0 + C_1t + C_2t^2 + C_3t^3 + C_4t^4 + C_5t^5 + C_6t^6] = 0,$$

with definite expressions for $A_0, A_1, \ldots, B_1, \ldots$ and $C_1, \ldots$.

5. The Solution

The solution of the differential equation (37), avoiding the singularity is written as

(38) $$u = \sum_{n=0}^{\infty} a_n\, t^{n+m/2}$$

which confirms that $m = 0$ and $m = 1$ should be disregarded.

Substituting the series (38) in (37) we get a seven term recurrence relation between the coefficients of the series

(39) $$a_n\,[\,4m^2\{A_0\,((n + \tfrac{m}{2})\,(n + \tfrac{m}{2} - 1) + (n + \tfrac{m}{2}))\} - m^4A_0] +$$

$$+ a_{n-1}\,[\,4\,\{A_1\,(n + \tfrac{m}{2} - 1)\,(n + \tfrac{m}{2} - 2) + B_1(n + \tfrac{m}{2} - 1)\} + C_1] +$$

$$+ a_{n-2}\,[\,4\,\{A_2\,(n + \tfrac{m}{2} - 2)\,(n + \tfrac{m}{2} - 3) + B_2(n + \tfrac{m}{2} - 2)\} + C_2] +$$

$$+ a_{n-3}\,[\,4\,\{A_3\,(n + \tfrac{m}{2} - 3)\,(n + \tfrac{m}{2} - 4) + B_3(n + \tfrac{m}{2} - 3)\} + C_3] +$$

$$+ a_{n-4}\,[\,4\,\{A_4\,(n + \tfrac{m}{2} - 4)\,(n + \tfrac{m}{2} - 5) + B_4(n + \tfrac{m}{2} - 4)\} + C_4] +$$

$$+ a_{n-5}\,[\,4\,\{A_5\,(n + \tfrac{m}{2} - 5)\,(n + \tfrac{m}{2} - 6) + B_5(n + \tfrac{m}{2} - 5)\} + C_5] +$$

$$+ a_{n-6}\,C_6 = 0$$

Arranging in descending power of n and dividing by $n^2 a_{n-6}$ and letting.

(40) $$\underset{n \to \infty}{Lt} \frac{a_n}{a_{n-1}} = p,$$ and taking the limit as $n \to \infty$, we have

(41) $$f(p) = (m^2 A_0 p^5 + A_1 p^4 + A_2 p^3 + A_3 p^2 + A_4 p + A_5)$$

Now $f(0), f(1)$ and (-1) are

$$f(0) = A_5 = b^4 a^4 x^2,$$

$$f(1) = D^2(m^2 + x^2), \text{ for } a = 1,$$

$$f(-1) = (x^2 - m^2)(1 + 16 b^2 c^2 m^2 + 16 b^4 + 32 b^3 cm - 8 b^2 - 8 bcm) \text{ for } a = 1,$$

If $x < m$, we see that the roots lie between zero and -1 for $a = 1$.

6. The boundary conditions

The following should be continuous on the boundary of the cylinder at $r = R$. *i*) grade V, *ii*) $\overline{H} \cdot \overline{n}$, *iii*) $\overline{J} \cdot \overline{n}$ and *iv*) the normal component of the total stress tensor (material + electromagnetic),

$\overline{H}$ denotes the total magnetic field during the oscillations, $\overline{J}$ being the current and $\overline{n}$ the unit normal to the boundary outwards.

These conditions on S_0 (the boundary at equilibrium) at $r = R$, reduce to

(42) $$\text{grade} \triangle V_e - \text{grade} \triangle V_i = 4\pi G \rho \xi_r,$$

(43) $$\triangle V_1 = \triangle V_e$$

(44) $$\delta p + \tfrac{1}{4\pi}[H_z^2 ik \xi_z + KH_z im \xi_z + KH_z ikr \xi_\phi + k^2 imr \xi_\phi]$$

$$= \frac{\triangle |\overline{H}_e|^2}{8\pi} + \frac{1}{8\pi}(\overline{\xi} \cdot \text{grad}) | H_e |^2,$$

where δ and $\triangle$ respectively denote Lagrangian and Eulerian perturbations.

The solution of equations (19) gives the perturbation in the vacuum magnetic field as

(45) $$[\triangle \overline{H}_e] = A_0 \left[k K_m'(k_r), \frac{im}{r} K_m(k_r), ik K_m(k_r) \right],$$

where A_0 is a constant of integration.

The continuty of the magnetic field at the surface of the cylinder determines the value of the constant A_0

(46) $$A_0 = \frac{1}{k K_m'(x)} [m K + kH_c(1 - a)].$$

The perturbation in the gravitational potential V_i, is given by [*cf.* (6) pp. 517

(47) $$\triangle V_i = -4\pi G \rho R \xi_R K_m(x) I_m(kr).$$

The boundary condition (44) with the help of equations (22), (45), (46) and (10) at $r = R$ on simplification reduces to the form

(43) $$2t \frac{du}{dt}[1 - 2 bcm(1 - a) - b^2(1 - a)^2] +$$

$$u\,[1-2\,bcm\,(1-a)-b^2\,(1-a)^2+(bcm+m^2c^2)\,2\,+$$
$$+\,(m^2+x^2)\,\{f(cm+b\,(1-a))^2-c^2+J\,(I_mK_m-\tfrac{1}{2})\}\,]=0.$$

with

(49) $$f=\frac{K_m}{x\,K_m'}\,,\;J=\frac{4\pi\rho G}{D}\,.$$

7. The Case II $k=0$

Equations (24–27) reduce to

(50) $$\sigma^2\xi_r=\frac{\partial\zeta}{\partial r}+\frac{1}{4\pi\rho}\left[\frac{\partial}{\partial r}\,(KH_z\,im\xi_z+K^2\,\,imr\,\xi_\phi)\right]$$
$$+\frac{K^2}{4\pi\rho}\,(m^2\xi_r+2im\,\xi_\phi)-\frac{\partial}{\partial r}\left[\frac{\xi_r\,.\,H_z}{4\,\pi\rho}\,\frac{\partial H_z}{\partial r}\right],$$

(51) $$\sigma^2\,\xi_\varphi=\frac{im\,\zeta}{r}-\frac{2K^2}{4\pi\rho}\,im\,\xi_r-\frac{i\,m}{r}\,\frac{H_z}{4\pi\rho}\,\frac{\partial H_z}{\partial r}-\frac{KH_z}{4\pi\rho}\,\frac{m^2}{r}\,\xi_z,$$

(52) $$\sigma^2\xi_z=\frac{K^2}{4\pi\rho}\,m^2\,\xi_z\,,$$

(53) $$(r\,\xi_r)'+im\,\xi_\phi=0.$$

Equations (52), (53) and (51) immediately give

(54) $$\xi_z=0,\quad\text{where }\sigma^2-m^2\,C^2\neq 0,$$

(55) $$\xi_\phi=\frac{i}{m}\,(r\,\xi_r)'\,,$$

and

(56) $$\zeta=\frac{\sigma^2}{m^2}\,r(r\,\xi_r)'+r\,\xi_r\,\left[2\,C^2-2B^2\left(1-\frac{ar^2}{R^2}\right)\right].$$

The differential equation for

$$u=r\,\xi_r\qquad\text{if }\sigma^2-m^2\,C^2\neq 0\text{ , is}$$

(57) $$\frac{\partial}{\partial r}\left(r\,\frac{du}{dr}\right)=m^2\,\xi_r\quad.$$

This is a homogenous differential equation and its solution avoiding singularity is written as

(58) $$\xi_r=A\,r^{m-1},$$

This solution of ξ_r again confirms that in this case also $m=0$ and $m=1$ are to be disregarded.

The solutions of ξ_ϕ and ζ are from (55) and (56)

(59) $$\xi_\phi=i\,A\,r^{m-1},$$

(60) $$\zeta=A\,r^m\left[\frac{\sigma^2}{m^2}+2\left\{C^2-B^2\left(1-\frac{ar^2}{R^2}\right)\right\}\right],$$

(61)
$$\triangle V_i = E\, r^m ,$$
$$\triangle V_e = E\, R^2\, r^{-m-2} ,$$
$$\frac{\triangle p}{\rho} = R^m\left[A \left\{\frac{\sigma^2}{m^2} + 2\left(C^2 - B^2 \overline{1 - \frac{ar^2}{R^2}}\right)\right\} - E \right],$$

where E is a constant.

The corresponding boundary condition when $k = 0$, at the surface of the cylinder $r = R$ determines the frequency of oscillation. Equation (44) reduces to

(62)
$$\frac{\triangle p}{\rho} + \frac{1}{4\pi\rho} K H_z\, im\xi_z = \frac{K^2\, imr\, \xi\phi}{4\pi\rho} = \frac{1}{4\pi\rho}\left[H^e_\phi (\triangle \overline{H}_e)_\phi + H^e_z (\triangle\, \overline{H}_e)_z \right] + \frac{1}{4\pi\rho}\, \xi_r\, H^e_\phi \frac{\partial H_\phi{}^e}{\partial r} .$$

Also in this case

(63)
$$(\triangle \overline{H}_e)_\phi = m\, k\, \xi_R \;, \; (\triangle\, H_e)_z = 0.$$

Thus equation (62) gives the frequency of oscillation as

(64)
$$\sigma^2 = m\left[\frac{E}{A} + C^2\,(m-1)\right.$$

This is always positive for $m > 0$ and depends upon the toroidal component of magnetic field. It is independent of poloidal component as should be the case. Thus in this case the oscillations are always stable and toroidal component increases the frequency of osillation.

8. The Electric field

The electric field inside is given by a well-known formula

(65)
$$E_i = -\frac{i\sigma}{c}\left(\overline{\xi} \times \overline{H}\right),$$

The Maxwell's equations for the vacuum are

(66)
$$\text{div}\ \overline{E}_e = 0 \text{ and curl } \overline{E}_e = -\frac{i\,\sigma}{c} \triangle \overline{H}_e.$$

The solution of equations (66) as obtained in paper[9] is

(67)
$$(E_e)_z = A\ K_m\,(kr) ,$$

(68)
$$(E_e)_\phi = \frac{m}{kr} A K_m\,(kr) + \frac{\sigma}{c} A_0\, K_m{}'\,(kr),$$

and

(69)
$$(E_e)_r = -iA\ K_m{}'\,(kr) - \frac{im\,\sigma}{ckr} A_0\, K_m(kr),$$

A and A_0 are constants to be determined from continuity conditions.

The continuty of ϕ and z components of $\overline{E}_e$ at the surface $r = R$ determines the two constants A and A_0

(70) $$A = \frac{K(u)_R}{K_m(x)}\,, \text{ and } A_0 = -\frac{ick}{K_m'(x)}\left[\frac{m\,u\,K}{x} + \frac{u\,H_0\,(1-a)}{R}\right]_{r=R}.$$

The corresponding electric field when $k = 0$ is (as $\triangle \overline{H}_e = 0$ when $k = 0$)

(71) $$\overline{E}_e = A_1 \text{ grad } [\, r^{-m} \exp\,[i(\sigma t + m\phi)]\,],$$

where, because of the continuity of the z-component

(72) $$A_1 = -\left[\frac{i\,K\,(u)}{k}\right]_{r=R} R^m\,,$$

where $(u)_R$ means the value of a $r\,\xi_r$ at $r = R$.

9. Conclusions

The coefficients of the delferential equation (37) depend upon b, c, m and x^2. In obtaining numerical solution the ratio c/b is taken to be unity and then for each fixed b and m we get a different series for every x. We have obtained solutions for $x = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$, $0{\cdot}4$ and $0{\cdot}5$ in every case. It is seen that the series are rapidly convergent. The parameter a occurring in the magnetic field (2) is chosen to be unity.

We have also considered the case when a is so small that its second and higher powers are neglected. Seven term recurrence relation reduces to that of three term relation. For numerical work it is chosen to be 0·01. Series are obtained for certain x^2 *viz.* 0·1, 0·25, 0·36 etc.

In each case the value of $J = \dfrac{4\pi G\rho}{D}$ has been evaluated from the boundary conditions (48). The displacement function $(r\xi_r)$ has also been calculated. These calculations are made for $m = 2$ and are given in tables (1-4).

TABLE I

The (x^2, u, J) relation for $b^2 = 0{\cdot}005$ and $a = 1$

x^2	u	$J = \dfrac{4\pi G\rho}{D}$
0·01	0·996137	3·022313
0·04	1·000979	2·994833
0·09	1·009095	2·951012
0·16	1·020491	2·892548
0·25	1·057055	2·857905

The corresponding series for $x^2 = 0{\cdot}01$ is $u = t - 0{\cdot}00521534t^2 + 0{\cdot}0013907t^3 - 0{\cdot}000043953t^4 + 0{\cdot}00005383t^5 - \ldots\ldots$

for $x^2 = 0{\cdot}04$,

$$u = t - 0{\cdot}000348543t^2 + 0{\cdot}00135764t^3 - 0{\cdot}000034528t^4 + 0{\cdot}0000049146t^5 - \ldots\ldots,$$

for $x^2 = 0{\cdot}09$,

$$u = t + 0{\cdot}007762787t^2 + 0{\cdot}0013326t^3 - 0{\cdot}000001245t^4 + 0{\cdot}0000003380t^5 - \ldots\ldots,$$

for $x^2 = 0{\cdot}16$,

$$u = t + 0{\cdot}01911865t^2 + 0{\cdot}00136744t^3 + 0{\cdot}0000016889t^4 + 0{\cdot}0000003482t^5 - \ldots\ldots$$

and for $x^2 = 0{\cdot}25$,

$$u = t + 0{\cdot}05563445t^2 + 0{\cdot}00130756t^3 + 0{\cdot}00011496t^4 - 0{\cdot}000002317t^5 + \ldots\ldots$$

TABLE II

The (x^2, u, J) relation for $m = 2$, $a = 1$, and $b^2 = 0{\cdot}01 = c^2$

x^2	u	$J = \dfrac{4\pi G\rho}{D}$
0·01	0·992693	3·057925
0·04	0·995153	3·025327
0·09	1·002960	2·979863
0·16	1·013953	2·919103
0 25	1·028194	2·845083

The corresponding series for $x^2 = 0{\cdot}01$ is $u = t - 0{\cdot}0101348t^2 + 0{\cdot}002983t^3 - 0{\cdot}0001797t^4 + 0{\cdot}00002683t^5 - 0{\cdot}000002592t^6 + \ldots\ldots,$
for $x^2 = 0{\cdot}04$,

$$u = t - 0{\cdot}00774736t^2 + 0{\cdot}00501t^3 - 0{\cdot}000171224t^4 + 0{\cdot}000024t^4 - 0{\cdot}00000223t^6 + \ldots,$$

for $x^2 = 0{\cdot}09$,

$$u = t + 0{\cdot}00011228t^2 + 0{\cdot}0029797t^3 - 0{\cdot}00015666t^4 + 0{\cdot}0000025497t^5 - \ldots\ldots,$$

for $x^2 = 0{\cdot}16$,

$$u = t + 0{\cdot}0111158t^2 + 0{\cdot}0029306t^3 - 0{\cdot}00011637t^4 + 0{\cdot}00025134t^5 \ldots\ldots$$

and for $x^2 = 0{\cdot}25$,

$$u = t + 0{\cdot}25263t^2 + 0{\cdot}002974t^3 - 0{\cdot}000065917t^4 + 0{\cdot}000028053t^5 - 0{\cdot}0000050843t^6 + \ldots\ldots$$

TABLE III

The (x^2, u, J) relation for $b^2 = 0{\cdot}02$, $m = 2$

x^2	u	$J = \dfrac{4\pi G\rho}{D}$
0·01	0·978162	3·112320
0·04	0·983082	3·086155
0·09	0·990312	3·037957
0·16	1·000401	2·971358
0·25	1·013503	2·891444

The corresponding series for $x^2 = 0{\cdot}01$ is $u = t - 0{\cdot}0279934t^2 + 0{\cdot}00753929t^3 - 0{\cdot}0018672t^4 + 0{\cdot}00080329t^5 - 0{\cdot}00032038t^6 + \ldots\ldots,$
for $x^2 = 0{\cdot}04$;

$$u = t - 0{\cdot}0238288t^2 + 0{\cdot}00775987t^3 - 0{\cdot}00102075t^4 + 0{\cdot}000210308t^5 - 0{\cdot}00003863t^6 + \ldots\ldots,$$

for $x^2 = 0{\cdot}09$,

$$u = t - 0{\cdot}0165778t^2 + 0{\cdot}0076631t^3 - 0{\cdot}00098207t^4 + 0{\cdot}00024139t^5 - 0{\cdot}00003231t^6 + \ldots\ldots,$$

for $x^2 = 0{\cdot}16$,

$$u = t - 0{\cdot}00642634t^2 + 0{\cdot}00758396t^3 - 0{\cdot}0009298t^4 + 0{\cdot}00020106t^5 - 0{\cdot}0000278211t^6 + \ldots\ldots,$$

for $x^2 = 0{\cdot}25$,

$$u = t - 0{\cdot}0066255t^2 + 0{\cdot}0075750t^3 - 0{\cdot}000864165t^4 + 0{\cdot}000210508t^5 - 0{\cdot}00004386t^6 + \ldots\ldots$$

TABLE IV

The (x^2, u, J) relation for $a = 0{\cdot}01$ and $b^2 = 0{\cdot}1$

x^2	u	J
0·25	0·935405	1·917778
1·0	0·764508	1·153716

The corresponding series for $x^2 = 0{\cdot}25$ and $x^2 = 1$ respectively are

$$u = t - 0{\cdot}0658333t^2 + 0{\cdot}00127041t^3 - 0{\cdot}0000269167t^4 - 0{\cdot}0000004981t^5 + \ldots\ldots\ldots$$

and

$$u = t - 0{\cdot}255833t^2 + 0{\cdot}02046776t^3 - 0{\cdot}000023465t^4 - 0{\cdot}0001032t^5 + \ldots\ldots$$

TABLE V

The (x^2 u, J) relation for $b^2 = 0{\cdot}025$ $a = 0{\cdot}01$

x^2	u	J
0·16	1·024693	2·682035
0·25	1·034831	2·593084

The corresponding series are $u = t + 0{\cdot}02437007t^2 + 0{\cdot}00032t^3 + 0{\cdot}00000247^4 + 0{\cdot}00000000907t^5 - \ldots\ldots\ldots$

$$u = t + 0{\cdot}0340408t^2 + 0{\cdot}0007792t^3 + 0{\cdot}000010615t^4 + 0{\cdot}00000009433t^5 + \ldots\ldots$$

TABLE VI

For $b^2 = 0{\cdot}036$, $a = 0{\cdot}01$

x^2	u	J
0·25	1·029465	2·499900
0·36	1·043195	2·377654

The corresponding series are $u = t + 0{\cdot}288832t^2 + 0{\cdot}00057495t^3 + 0{\cdot}000064754t^4 + 0{\cdot}00000003477t^5 + \ldots\ldots\ldots$

$$u = t + 0{\cdot}0418494t^2 + 0{\cdot}001321517t^3 + 0{\cdot}00002366t^4 + 0{\cdot}000000027159t^5 + \ldots\ldots$$

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Kinetics of oxidation of D–glucose and D–xylose with vanadium (V) in acid medium

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Abstract

Kinetics of the oxidation of reducing sugars has been described with vanadium (V) in acidic media. It has been observed that the reaction is first order both with respect to vanadium (V) and reducing sugars. Logarithms of first order constant linearly depend on the concentrations of acids. Energy of activation of D–glucose and D–xylose are 22·95 and 16·79 K. Cals. respectively. Further, positive salt effect has also been observed on the rate of reaction.

Introduction

Littler and Waters have recently studied the mechanism of the oxidation of pinacol[1] by vanadium (V) where they reported the reaction to be first order both with respect to vanadium (V) and pinacol. In course of their kinetic study they have reported small linear salt effect and complex dependence of acids. During the kinetic study of the oxidation of cyclohexan-one and cycloheptanone[2] by sulphuric and perchloric acids they have reported that the reaction is first order both with respect to vanadium (V) and organic substrate. In these cases they have reported large positive salt effect and dependence of acid with complex nature. In the oxidation of cyclohexanol and 1–deutrocyclohexanol with vanadium (V) in sulphuric acid medium they have concluded the reaction to be first order both with respect to vanadium (V) and alcohols. They have mentioned that the oxidation of alcohols proceeds through complex states followed by a kinetic isotopic effect. The rate determining step involves C–H (or C–D) bond fission.

During the study of kinetics of the oxidation of tartaric acid by vanadium (V) in acidic medium Bakore and Bhargava[4] have reported that the reaction is first order with respect to vanadium (V) and tartaric acid and at low acid concentration the rate is independent to H^+ concentration but is directly proportional to H^+ concentration at higher acidity.

The object of our work is to find out whether β–D anomer is more active than α–D anomer as reported by Ingles and Israel[5] who have observed that β–D anomer is oxidised approximately 28 times faster than α–D anomer (pH about 9·2) by hypo=iodide ion and aqueous bromine. However the rate of mutarotation of free sugars is greatly enhanced in alkaline medium and beyond the pH range of 11·8 becomes much faster than oxidation, the result being that both forms are oxidized at the identical rate[6].

Further Bobtelsky and Glasner[7] have reported that the oxidising power of the cations of vanadium (V) increases as the acidity of the medium increases, in a manner indicating the existence of an equilibrium

$$VO_2^+ + H_3O = V(OH)_3^{2+}$$

This has also been supported by Waters and his co-workers (loc. cit.). Thus our object is to investigate these facts.

Present work deals with the kinetics of the oxidation of D-glucose and D-xylose with vanadium (V) in the presence of acids.

Experimental

Chemicals Employed : D-glucose (Bacto Difco grade sample) D-xylose (B. D. H. grade sample), sulphuric and hydrochoric acid (Analar B. D. H. grade sample) perchoric acid (A. R. Rediel grade sample), ferrous ammonium sulphate (Analar B. D. H. grade sample) potassium chloride (Analar B. D. H. grade sample), N-phenyl anthranilic acid (B. D. H. grade sample) and ceric sulphate (technical B. D. H. grade sample). Ceric sulphate (in 8N H_2SO_4) solution was standardised against a standard solution of ferrous ammonium sulphate using N phenyl anthranilic acid as an indicator. Sugar solutions were daily prepared by weighing exact quantity.

Kinetic measurements

Solutions of sodium vanadate and acid were kept for attaining constant temperature in the electrically regulated thermostat ($\pm$ 0·1°C). In a separate glass stoppered bottle a standard solution of reducing sugar was also kept for attaining constant temperature. The aliquots withdrawn at different intervals of time were quenched with an excess of a standard solution of ferrous ammonium sulphate. Unused ferrous was titrated against a standard solution of ceric sulphate using N-phenyl anthranilic acid as indicator.

Stoichiometry of reaction

The mixtures in different ratio of concentrations such that overall concentration of sodium vanadate was twenty times more than the overall concentrations of reducing sugars were prepared and kept for a number of days. The excess of unreacted vanadium (V) was then volumetrically estimated on different days. The maximum values of equivalence were 12 and 10 in case of D-glucose and D-xylose respectively 12 Equivalence may be explained by the following equation

$$C_6H_{12}O_6 + 12V^{+5} + 6H_2O \rightarrow 6HCOOH + 12H^+ + 12V^{+4}$$

whereas 10 equivalence obtained in case of D-xylose may be explained as

$$C_5H_{10}O_5 = 10V^{+5} + 5H_2O \rightarrow 5HCOOH + 10H^+ + 10V^{+4}$$

The stoichiometric results show that the main product of the reaction is formic acid.

Results and Discussions

Determination of the order of the reaction with respect to vanadium (V) : In order to find out the order of reaction with respect to vanadium (V), the concentrations of reducing sugars and the acid were taken in large excess as shown table 2. It has been observed that for typical runs plots of log (a—x) and time are straight lines showing that order of the reaction is first with respect to vanadium (V) (fig. 1). The slight deviation from the straight line in the latter part of the r eaction may be due to further oxidation of the reaction product. Although the rate at which vanadium (V), disappears follows first order rate law for individual

runs, the values of the rate constants increase with the increase in the concentration of vanadium (V) in the range from 0·00167 to 0·04 M (table 1A and B for D-glucose and D-xylose respectively). Similar results are also obtained at constant ionic strength (table 1C.) The reason for this increase is not clear at present.

TABLE 1A

Effect of vanadium (V) variation on the rate of oxidation of D-glucose.

D-glucose = 0·25 M. H_2SO_4 = 6N Temperature 40°C

$NaVO_3$(M)	0·04	0·02	0·01	0·008	0·006	0·005	0·0025	0·002	0·00167
$k_1 \times 10^3$ min^{-1}	4·16	3·38	3·34	3·18	3·04	3·07	3·08	3·32	3·45

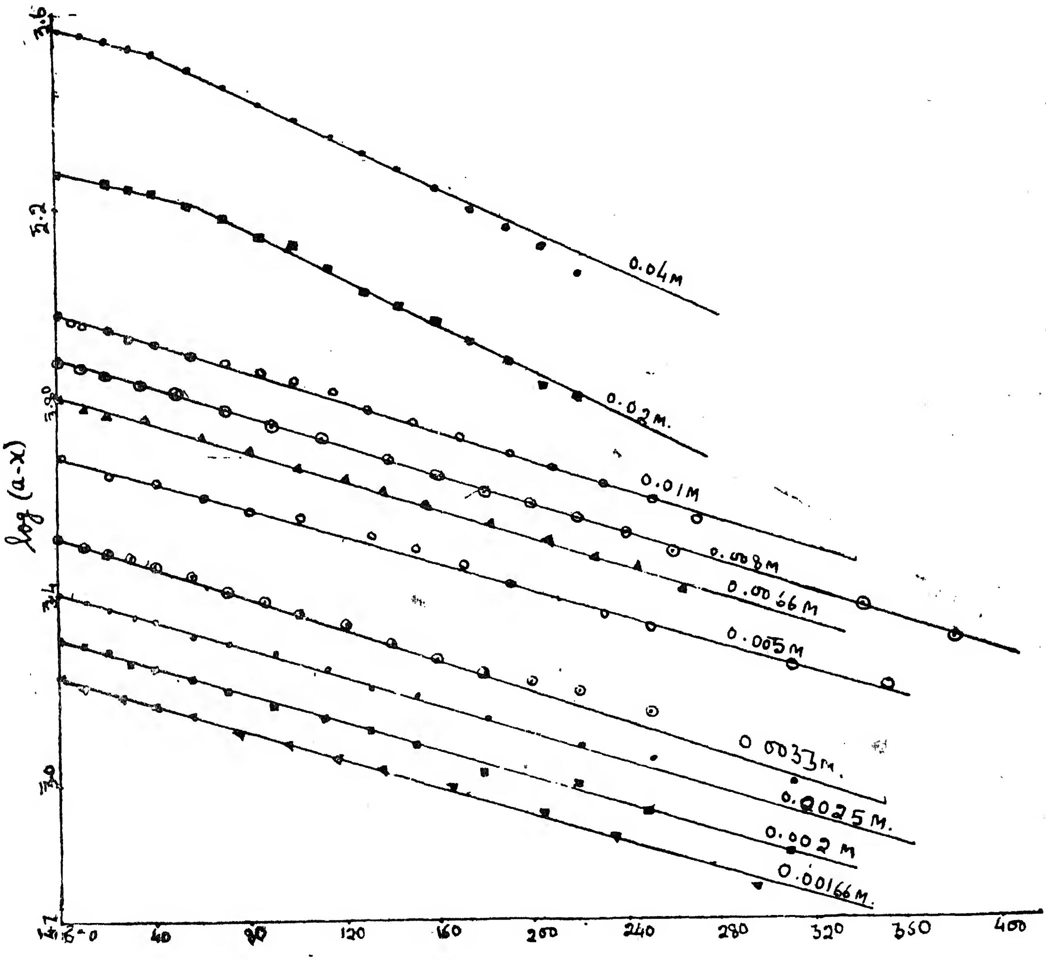


Fig. 1. Variation of log $(a-x)$ and T for Vanadium (V)., concentration of Vanadium (V) taken are written on the curve.
D-glucose = 0·25 M, H_2SO_4= 6 N, Temprature = 40°C.

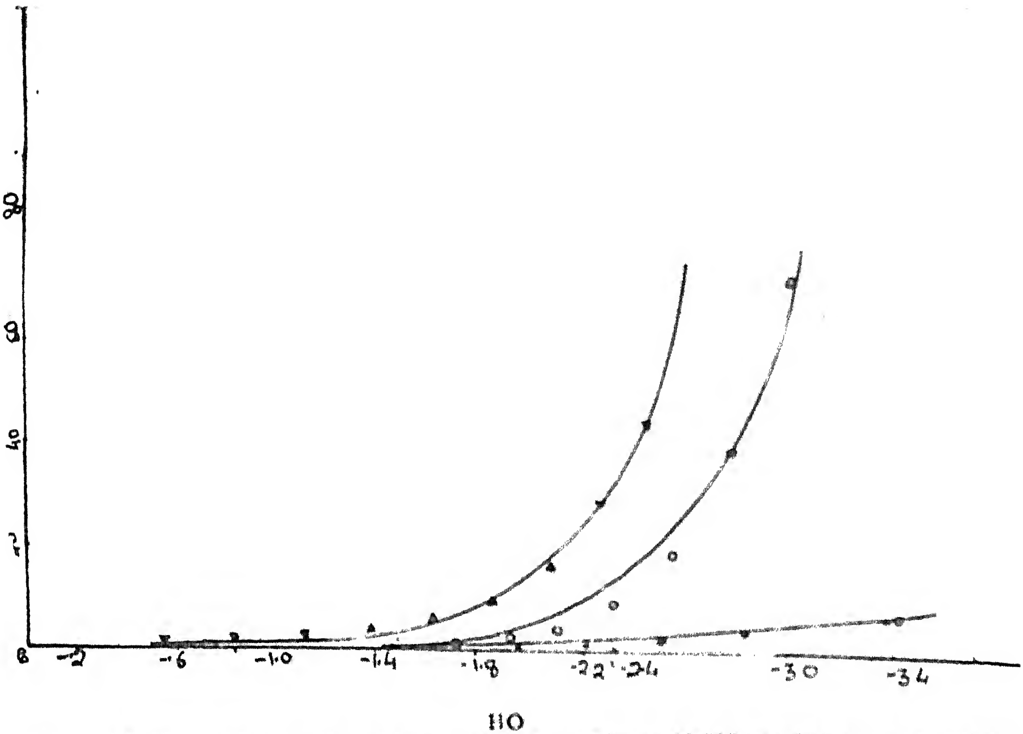
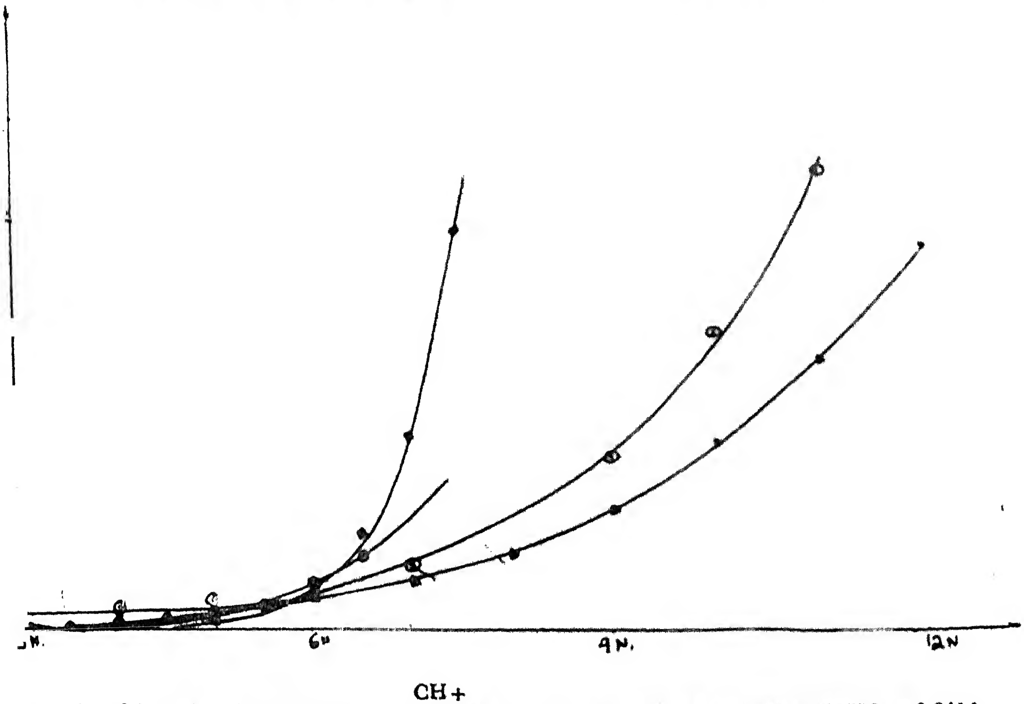


Fig. 2. Variation of k_1 with HO of Acids Temprature=25°C. $NaVO_3$=0·01 M. Sugar =·25M.
▼ = D-xylose in H_2SO_4 O = D-glucose in HCl. ● = D-glucose in $HClO_4$



3. Plot of k_1 against Hydrogen ion concentration. Reducing Sugar = 0·25M, $NaVO_3$ = 0·01M
D–glucose and HCl at 25°C. O = D–glucose and $HClO_4$ at 25°C

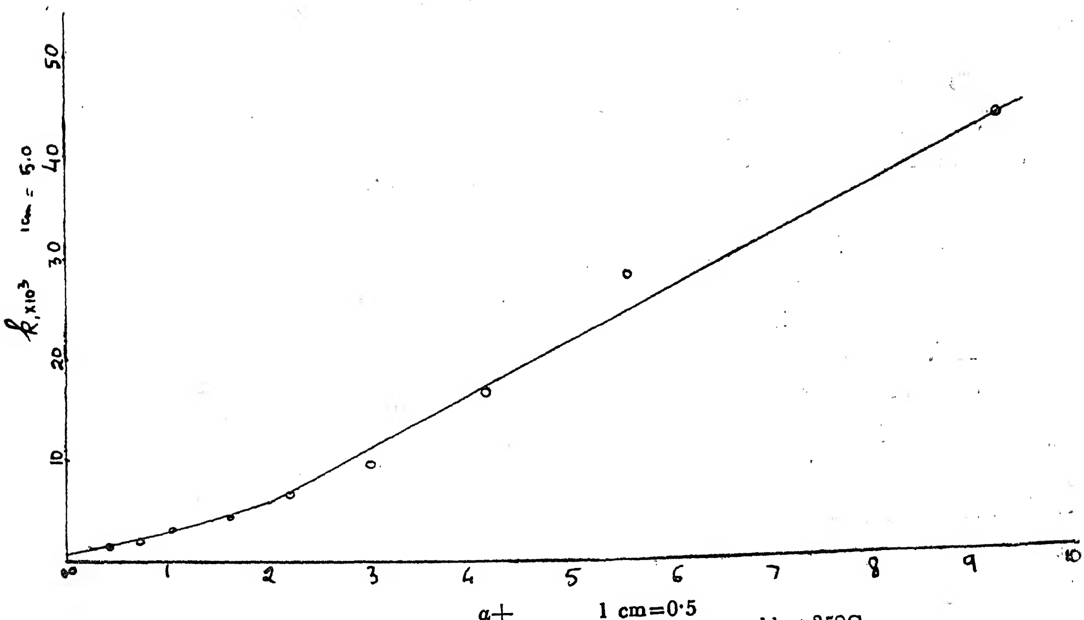


Fig. 4. Variation of k_1 with total activity of Sulphuric acid at 25°C.
D-xylose=·25 M, $NaVO_3$=0·01 M

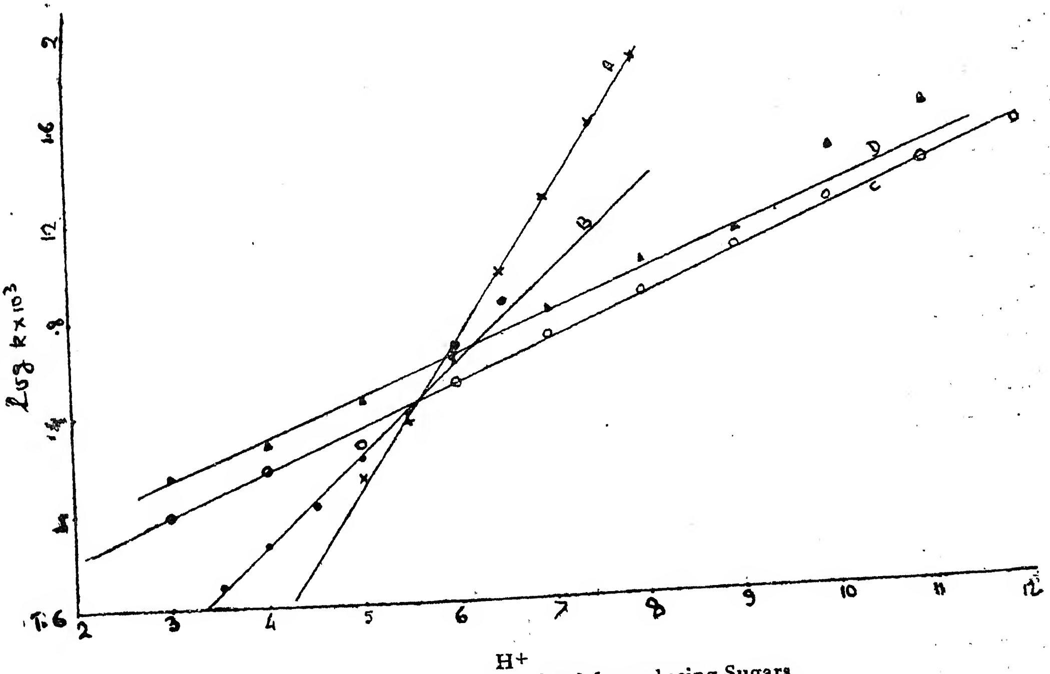


Fig. 5. Plot of H+ ion concentration against log k for reducing Sugars.
$NaVO_3$=·01M. Reducing Sugar=·25 M
A. D-glucose and HCl at 25°C. B. D-glucose and $HClO_4$ at 25°C. C. D-glucose and H_2SO_4 at 40°C.
D, D-xylose and H_2SO_4 at 25°C

TABLE 1B

Effect of variation of vanadium (V) on the rate of oxidation of D-xylose

D-xylose = 0·1 M. H_2SO_4 = 6N Temperature 40°C

$NaVO_3$(M)	0·04	0·02	0·01	0·005	0·0033	0·0025	0·00166
$k_1 \times 10^3$ min^{-1}	13·4	13·0	8·81	8·21	9·10	9·20	10·4

TABLE 1C

Effect of variation of sodium vanadate on the rate of oxidation of D-glucose Ionic strength is kept constant by Potassium chloride

D-glucose = 0·25 M. H_2SO_4 = 6N Temperature 40°C

$NaVO_3$(M)	·04	·02	·01	·005
$k_1 \times 10^3$ min^{-1}	4·16	3·68	3·29	3·07

Determination of order with respect to Reducing Sugars : To determine the order of reaction with respect to reducing sugars concentrations of D glucose and D-xylose were varied from 0·5 to 0·125M. and 0·25 to 0·025 M respectively. It is observed that the first order constants gradually decrease in proportion on the reducing sugar in table 2. $\frac{k}{\text{reducing sugar}}$ calculated are practically constant. It is obvious that under identical conditions of other reactions the rate of oxidation of D-xylose is approximately 10 times faster than D-glucose.

Effect of change of hydrogen ion on the reaction rate : In table 3 the effect of the change of hydrogen ion concentration on the reaction rate using, sulphuric, hydrochloric and perchloric acids in case of D-glucose (table 3A, B, C) and sulphuric acid in case of D-xylose (table 3D) has been shown. Reaction rate has also been studied by keeping ionic strength constant with sodium perchlorate in perchloric acid (table 4). As acid concentrations employed are very large, the mean first order constants are plotted against acidity function H_0 and H_3O^+ concentrations. However the oxidation is not SO simple for the rate of reaction markedly increases at high acidities as fig. 2 and 3 show. H_0 was obtained from figures of Paul and Long[8]. To elucidate this effect of strong acids the oxidation rate has been plotted against various acidities functions. Also, the total rate constant k has been divided into two portions k_0 a hydrogen independent value obtained by extrapolating the measured rate to H_3O^+ = O and k_1 and dependent portion such that $k = k_0 + k_1$ when total activity of acid ($a \pm$) is plotted against first order constants the curve appears to be straight line in the region where acid dependent reaction is most important (fig. 4). The values of $a \pm$ have been taken from figures of Stokes[9]. Plot of log k_1 against H_3O^+ gives a straight line (fig. 5) which show that k_1 is exponentially proportional to acid concentration.

TABLE 2A

Effect of variation of D-glucose on the reaction rate

H_2SO_4 = 6N. $NaVO_3$ = 0·01 M. Temperature 40°C

D-glucose (M)	0·5	0·375	0·25	0·125
$k_1 \times 10^3$ min^{-1}	6·74	5·05	3·34	1·58
$\frac{k_1 \times 10^3}{\text{D-glucose}}$	13·5	13·5	13·4	12·7

TABLE 2B

Effect of variation of D–xylose on the reaction rate

$H_2SO_4 = 6N$ $NaVO_3 = 0{\cdot}01$ M. Temperature 40°C

D-xylose (M)	0·25	0·2	0·1	0·05	0·033	0·025
$k_1 \times 10^2$ min	3·0	1·84	·986	·66	·471	·405
$\frac{k_1 \times 10^2}{\text{xylose}}$	12·0	9·2	9·86	13·2	14·13	16·2

TABLE 3A

Effect of Sulphuric acid on the reaction rate of D–glucose

D–glucose = 0·25 M. $NaVO_3 = 0{\cdot}01$ M. Temperature 40°C

H_2SO_4(N)	3	4	5	6	7	8	9	10	11	12
$k_1 \times 10^3$ min^{-1}	0·96	1·48	1·83	3·34	4·96	7·37	11·55	17·85	25·61	36·65

TABLE 3B

Effect of Perchloric acid variation on the reaction rate of D–glucose

D–glucose = 0·25 M. $NaVO_3 = 0{\cdot}01$ M. Temperature 25°C

Perchloric acid (N)	3· 5	4	4·5	5·	5·5	6	6·5
$k_1 \times 10^3$ min^{-1}	·493	·73	1·06	1·59	2·48	4·03	7·09
H_0	–1·47	–1·72	–1·97	–2·23	–2·52	–2·84	–3·22

TABLE 3C

Effect of Perchloric acid uariation on the reaction rate of D–glucose

D–glucose = 0·25 M. $NaVO_3 = 0{\cdot}01$ M. Temperature 25°C

HCl(N)	5	5·5	6	6·5	7	7·5	8
$k_1 \times 10^3$ min^{-1}	1·32	2·34	4·12	9·30	18·91	38·5	71·45
H_0	–1·76	–1·93	–2·12	–2·34	–2·56	–2·78	–3·00

TABLE 3D

Effect of Sulphuric acid variation on reaction rate of D–xylase

D–xylose = 0·25 M. $NaVO_3 = 0{\cdot}01$ M. Temperature 25°C

H_2SO_4(N)	3	4	5	6	7	8	9	10	11
$k_1 \times 10^3$ min^{-1}	1·36	1·89	2·82	4·21	6·36	9·93	16·5	28·5	43·9
H_0	–0·56	–0·84	–1·12	–1·38	–1·62	–1·85	–2·06	–2·28	–2·51
$a_\pm$	0·4266	0·68	1·0405	1·5402	2·2181	3·088	4·203	5·59	9·24

TABLE 4

Effect of variation of 4 Perchloric acid. Ionic strength kept constant by sodium perchlorate = 4·51

D–glucose = 0·25 M. $NaVO_3 = 0{\cdot}01$ M. Temperature 40°C

$HClO_4$(N)	4·5	4·0	3·5
$k_1 \times 10^3$ min	7·57	5·24	3·84

TABLE 5

(A)

Effect of temperature on the reaction rate

Glucose = 0·25 M. $NaVO_3$ = 0·01 N H_2SO_4 = 7N

Temperature °C	35	40	45	50
$k_1 \times 10^3$ min^{-1}	2·76	4·96	8·69	15·3

(B)

H_2SO_4 = 6 N D-xylose = 0·25 M. $NaVO_3$ = 0·01 N

Temperature °C	25	30	35	40
$k \times 10^3$ min^{-1}	4·21	7·60	13·8	30·0

Effect of temperature on the reaction rate : The reaction rate has been studied at four different temperatures. Log k_1 has been plotted against $\frac{1}{T}$ where T is in Kalvin which is a straight line. Values of energy of activation calculated from the slope in the case of D-glucose and D-xylose are 22·95 and 16·79 K. Cals. respectively (Table 5).

As the reaction is first order both with respect to vanadium (V) and the reducing sugars, also approximately independent of hydrogen ion in lower concentrations of the acids, the following rate determining step has been postulated: VO_2 + reducing sugar $\xrightarrow{k}$ V^{4+} + other reaction products. Now at higher concentrations of the acids the reaction rate increases with the concentration of hydrogen ion. It appears that the reacting species of the vanadium is $V(OH)_3^{2+}$ as postulated by Bobtelsky and Glasner supported by Waters (loc. cit.). Thus under such conditions the following rate determining step is probable.

$$V(OH)_3^{2+} + \text{reducing sugar} \xrightarrow{k} V^{4+} + \text{other reaction products.}$$

From this study it also appears that the rate of oxidation of both the anomers α and β are approximately equal under the conditions of acid concentrations. Work is in progress and detailed study of the reaction products and other things will be published in the next series of this publication.

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Integrals Involving Products of G–Function

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The main object of this note is to find a formula in order to evaluate definite integrals involving the product of three Meijers G–function[5] in terms of S–function defined by Sharma[7]. This function is a generalisation of hypergeometric functions of two variables defined by Appellet Kampe' de Feriet[1]. Since G–function is the generalization of a great many of the special functions the formulae can yield many results involving Bessel, Legendre, Whittaker functions and other related functions. Integrals proved recently by Bailey[4] and Sharma[8] follow as special case of our findings. In what follows n and m are positive integers and $\triangle(s, a)$ represents the set of parametres

$$\frac{a}{s}, \frac{a+1}{s}, \ldots, \frac{a+s-1}{s} \text{ and}$$

$$[\triangle(s, a_r)] = \triangle(s, a_1), \triangle(s, a_2), \ldots, \triangle(s, a_r)$$

The result to be established here is

$$\int_0^\infty G_{C,\,D}^{A,\,B}\left(ax \,\middle|\, \begin{matrix} e_1, \ldots, e_c \\ f_1, \ldots, f_D \end{matrix}\right) G_{q,\,r}^{h,\,o}\left(bx \,\middle|\, \begin{matrix} \alpha_1, \ldots, \alpha_q \\ \beta_1, \ldots, \beta_r \end{matrix}\right) G_{\gamma,\,\delta}^{\alpha,\,\beta}\left(cx^{\frac{n}{m}} \,\middle|\, \begin{matrix} a_1, \ldots, a_\gamma \\ b_1, \ldots, b_\delta \end{matrix}\right) dx$$

$$= (2\pi)^{(1-n)(h-\frac{1}{2}q-\frac{1}{2}r)}\, n^{\Sigma\beta_j - \Sigma\alpha_j + \frac{1}{2}(r-q)}\, (2\pi)^{(1-m)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)}$$

$$\times\, m^{\Sigma b_j - \Sigma a_j + \frac{1}{2}\gamma - \frac{1}{2}\delta + 1}\, (2\pi)^{(1-n)(A+B-\frac{1}{2}C-\frac{1}{2}D)}\, n^{\Sigma f_j - \Sigma e_j + \frac{1}{2}C - \frac{1}{2}D + 1}$$

$$\times\, b^{-1}\, S\left[\begin{matrix} \begin{bmatrix} nh, & 0 \\ n(r-h), & nq \end{bmatrix} \\ \begin{bmatrix} nB, & nA \\ n(C-B), & n(D-A) \end{bmatrix} \\ \begin{bmatrix} m\beta, & m\alpha \\ m(\gamma-\beta), & m(\delta-a) \end{bmatrix} \end{matrix} \,\middle|\, \begin{matrix} [\triangle(n, \beta_r+1); [\triangle(n, \alpha_q+1)] \\ \\ [\triangle(n, l_c)]; (\triangle(n, f_D)] \\ \\ [\triangle(m, a_\gamma)]; [\triangle(m, b_\delta)] \end{matrix} \,\middle|\, \begin{matrix} \dfrac{a^n n^{n(c-D)}}{b_n n^{n(q-r)}} \\ \\ \dfrac{c^m m^{m(\gamma-\delta)}}{b^n n^{n(q-r)}} \end{matrix}\right]$$

When the following set of conditions (A) and (B) are satisfied

(A) The non negative integers A, B, C, D, h, q, r, α, β, γ, δ satisfy the following inequilities

$$q > 0,\ r > 0,\ D \geq 1,\ 0 \leqslant A \leqslant D,\ 0 \leqslant h \leqslant r,\ 0 \leqslant B \leqslant C,\ 0 \leqslant \beta \leqslant \gamma,$$
$$0 \leqslant \alpha \leqslant \delta,\ 2(A+B) > C+D,\ 2h > q+r,\ 2(\alpha+\beta) > \gamma+\delta,$$
$$|\arg a| < (A+B-\tfrac{1}{2}C-\tfrac{1}{2}D)\pi,\ |\arg b| < (h-\tfrac{1}{2}q-\tfrac{1}{2}r)\pi,\ |\arg c| < (\alpha+\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta)\pi$$

(B) $|\arg a/b| < (h+B+A-\frac{1}{2}q-\frac{1}{2}D-\frac{1}{2}r-\frac{1}{2}C)\pi,\ 2(h+B+A) > q+D+r+C,$

$|\arg c^m/b^n| < (nh+m\beta+m\alpha-\frac{1}{2}nq-\frac{1}{2}m\delta-\frac{1}{2}nr-\frac{1}{2}m\gamma)\pi,$

$2(nh+m\beta+m\alpha) > nq+m\delta+nr+m\gamma,$

$R\{\min f_j + \min \beta_j + n/m\, b_j + 1\} > 0 > R\{\max e_j + n/m \max a_j - n/m - 1\}$

2. For establishing the above result we shall need the following formulae [[4] pages 207, 209, 219]

(i)
$$G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\, x^s\, ds \tag{2}$$

(ii)
$$x^{\sigma} G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_r + \sigma \\ b_s + \sigma \end{matrix}\right) \tag{3}$$

(iii)
$$G^{4,0}_{2,4}\left(x \,\middle|\, \begin{matrix} a, a+\frac{1}{2} \\ a+b, a+c, a-c, a+b \end{matrix}\right) = 2\pi^{-\frac{1}{2}} x^{a} K_{b+c}(x^{\frac{1}{2}}) K_{b-c}(x^{\frac{1}{2}}) \tag{4}$$

(iv)
$$\Gamma n x = (2\pi)^{\frac{1}{2}-\frac{1}{2}n}\, n^{nx-\frac{1}{2}} \prod_{r=0}^{n-1}\Gamma\left(x+\frac{r}{n}\right)$$

from which it can be derived that, for positive integer n,

(v)
$$\Gamma(a+nx) = (2\pi)^{\frac{1}{2}-\frac{1}{2}n}\, n^{a+nx-\frac{1}{2}} \prod_{r=0}^{n-1}\Gamma\left(\frac{a+r}{n}+x\right) \tag{5}$$

and

(vi)
$$\Gamma(a-nx) = (2\pi)^{\frac{1}{2}-\frac{1}{2}n}\, n^{a-nx-\frac{1}{2}} \prod_{r=0}^{n-1}\Gamma\left(\frac{a+r}{n}-x\right) \tag{6}$$

(vii)
$$\int_0^{\infty} x^{\sigma-1} G^{h,l}_{q,r}\left(x \,\middle|\, \begin{matrix} \alpha_1, \ldots, \alpha_q \\ \beta_1, \ldots, \beta_r \end{matrix}\right) G^{\alpha,\beta}_{\gamma,\delta}\left(z x^{n/m} \,\middle|\, \begin{matrix} a_1, \ldots, a_\gamma \\ b_1, \ldots, b_\delta \end{matrix}\right) dx$$
$$= (2\pi)^{(1-n)(h+l-\frac{1}{2}q-\frac{1}{2}r)}\, n^{\Sigma\beta_j-\Sigma\alpha_j+(\sigma-\frac{1}{2})(r-q)}\,(2\pi)^{(1-m)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)}$$
$$\times\, m^{\Sigma b_j-\Sigma a_j+\frac{1}{2}\gamma-\frac{1}{2}\delta+1}\, G^{m\alpha+nl,\, m\beta+nh}_{m\gamma+nr,\, m\delta+nq}\left(\frac{z^m m^{m(\gamma-\delta)}}{n^{n(q-r)}} \,\middle|\, \begin{matrix} [\Delta(m, a_\beta)], [\Delta(n, 1-\beta_r-\sigma)], \Delta(m, a_{\beta+1}), \ldots, \Delta(m, a_\gamma) \\ [\Delta(m, b_\alpha)], [\Delta(n, 1-\alpha_q-\sigma)], \Delta(m, b_{\alpha+1}), \ldots, \Delta(m, b_\delta) \end{matrix}\right) \tag{7}$$

under appropriate conditions of covergence given with the integral.

$$(viii)\qquad S(x,y) = S\left[\begin{array}{l|l|l} \begin{bmatrix} m_1, 0 \\ p_1-m_1, q_1 \end{bmatrix} & a_1,\ldots,a_{p_1};b_1,\ldots,b_{q_1} & \\ \begin{bmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2 \end{bmatrix} & c_1,\ldots,c_{p_2};d_1,\ldots,d_{q_2} & x \\ \begin{bmatrix} m_3, n_3 \\ p_3-m_3, q_3-n_3 \end{bmatrix} & e_1,\ldots,e_{p_3};f_1,\ldots,f_{q_3} & y \end{array}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m_1}\Gamma(a_j+s+t)\prod_{j=1}^{m_2}\Gamma(1-c_j+s)\prod_{j=1}^{n_2}\Gamma(\alpha_j-s)}{\prod_{j=m_1+1}^{p_1}\Gamma(1-a_j-s-t)\prod_{j=1}^{q_1}\Gamma(b_j+s+t)\prod_{j=m_2+1}^{p_2}\Gamma(c_j-s)}\times$$

$$\times\frac{\prod_{j=1}^{m_3}\Gamma(1-e_j+t)\prod_{j=1}^{n_3}\Gamma(f_j-t)}{\prod_{j=n_2+1}^{q_2}\Gamma(1-d_j+s)\prod_{j=m_3+1}^{p_3}\Gamma(e_j-t)\prod_{j=n_3+1}^{q_3}\Gamma(1-f_j+t)}\,x^s y^t\,ds\,dt$$

$$(ix)\quad S(x,y) = \sum_{u=1}^{n_2}\sum_{v=1}^{n_3}(x)^{d_u}(y)^{f_v}\frac{\prod_{j=1}^{m_2}\Gamma(1-c_j+d_u)\prod_{j=1}^{n_2}{}'\,\Gamma(d_j-d_u)}{\prod_{j=m_2+1}^{p_2}\Gamma(c_j-d_u)\prod_{j=n_2+1}^{q_2}\Gamma(1-d_j+d_u)}$$

$$\times\frac{\prod_{j=1}^{m_1}\Gamma(a_j+d_u+f_v)\prod_{j=1}^{m_3}\Gamma(1-e_j+f_v)\prod_{j=1}^{n_3}{}'\,\Gamma(f_j-f_v)}{\prod_{j=m_3+1}^{p_3}\Gamma(e_j-f_v)\prod_{j=n_3+1}^{q_3}\Gamma(1-f_j+f_v)\prod_{j=1}^{q_1}\Gamma(1-f_j+f_v)\prod_{j=1}^{p_1}\Gamma(1-a_j-d_u-f_v)}\times$$

$$\times F\left[\begin{array}{l|l|l} p_1 & a_1+d_u+f_v,\ldots,a_{p_1}+d_u+f_v & x(-1)^{p_1+p_2-m_1-m_2-n_2} \\ p_2 & 1-c_1+d_u,\ldots,1-c_{p_2}+d_u & \\ p_3 & 1-e_1+f_v,\ldots,1-e_{p_3}+f_v & \\ q_1 & b_1+d_u+f_v,\ldots,b_{q_1}+d_u+f_v & y(-1)^{p_1+p_3-m_1\;m_3-n_3} \\ q_2-1 & 1-d_1+d_u,\ldots*\ldots 1-d_{q_2}+d_u & \\ q_3-1 & 1-f_1+f_v,\ldots*\ldots,1-f_{q_3}+f_v & \end{array}\right] \qquad (9)$$

Where the prime in π' indicates the omission of the factor of the type (d_j-d_j); the asterisk in the F denotes the omission of the parameter of the type $(1-f_j+f_j)$.

The above F-function represents the following double series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p_1}(a_j+d_u+f_v)_{m+n}\prod_{j=1}^{p_2}(1-e_j+d_u)_m\prod_{j=1}^{p_3}(1-e_j+f_v)_n}{\prod_{j=1}^{q_2}(1-d_j+d_u)_m\prod_{j=1}^{q_3}(1-f_j+f_v)_n\prod_{j=1}^{q_1}(b_j+d_u+f_v)_{m+n}}\times$$

$$\times\Big[x(-1)^{p_1+p_2-m_1-m_2-n_2}\Big]^m\Big[y(-1)^{p_1+p_3-m_1-m_3-n_3}\Big]^n\frac{1}{|m}\frac{1}{|n}. \tag{10}$$

Proof: If the integrand of (1) $G_{C,\,D}^{A,\,B}\left(ax\,\middle|\,\begin{matrix}e_1,\ldots,e_C\\ f_1,\ldots,f_D\end{matrix}\right)$ is replaced by its equivalent contour integral (2), then the order of integration be changed and the inner integral be evaluated with the help of (17); we get on using (2) again, the value of the integral as

$$(2\pi)^{(1-n)(h-\frac{1}{2}q-\frac{1}{2}r)}\,n^{\Sigma\beta_j-\Sigma\alpha_j+\frac{1}{2}(r-q)}\,(2\pi)^{(1-m)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)}$$

$$\times\, m^{\Sigma b_j-\Sigma a_j+\frac{1}{2}\gamma-\frac{1}{2}\delta+1}\,b^{-1}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{A}\Gamma(f_j-s)\prod_{j=1}^{B}\Gamma(1-e_j+s)}{\prod_{j=A+1}^{D}\Gamma(1-f_j+s)\prod_{j=B+1}^{C}\Gamma(e_j-s)}$$

$$\times\frac{\prod_{j=1}^{\alpha}\Big\{\Gamma\Big(\frac{b_j}{m}-t\Big)\ldots\Gamma\Big(\frac{b_j+m-1}{m}-t\Big)\Big\}\prod_{j=1}^{\beta}\Big\{\Gamma\Big(1-\frac{a_j}{m}+t\Big)\ldots\Gamma\Big(1-\frac{a_j+m-1}{m}+t\Big)\Big\}}{\prod_{j=\alpha+1}^{\delta}\Big\{\Gamma\Big(1-\frac{b_j}{m}+t\Big)\Gamma\Big(1-\frac{b_j+m-1}{m}+t\Big)\Big\}\prod_{j=1}^{q}\Big\{\Gamma\Big(\frac{1+\alpha_j}{n}+\frac{s}{n}+t\Big)\Gamma\Big(\frac{n+\alpha_j}{n}+\frac{s}{n}+t\Big)\Big\}}$$

$$\times\frac{\prod_{j=1}^{h}\Big\{\Gamma\Big(\frac{1+\beta_j}{n}+\frac{s}{n}+t\Big)\ldots\Gamma\Big(\frac{n+\beta_j}{n}+\frac{s}{n}+t\Big)\Big\}}{\prod_{j=h+1}^{r}\Big\{\Gamma\Big(-\frac{\beta_j}{n}-\frac{s}{n}+t\Big)\ldots\Gamma\Big(\frac{n-1-\beta_j}{n}-\frac{s}{n}-t\Big)\Big\}\prod_{j=\beta+1}^{\gamma}\Big\{\Gamma\Big(\frac{a_j}{m}-t\Big)\Gamma\Big(\frac{a_j+m-1}{m}-t\Big)\Big\}}$$

$$\times\left(\frac{a}{bn^{(q-r)}}\right)^s\left(\frac{c^m\,m^{m(\gamma-\delta)}}{b^n\,n^{n(q-r)}}\right)^t ds\,dt. \tag{11}$$

Putting s for s/n in (11), and applying the formulae (5) and (6) and then interpret it with the help of (8), we get (1).

The contour in the s-plane here is a straight line along the imaginary axis extending from $-i\infty$ to $i\infty$ with indentation, if necessary to ensure that the poles of $\Gamma(f_j-s), j=1, 2, \ldots, A$ are to the right of it and the poles of $\Gamma(1-e_j+s)$,

$j = 1, 2, \ldots, B$ and $\Gamma(1+\beta_j+s+nt), j = 1, 2, \ldots, h$ are to the left of the contour.

Similarly the contour in the t-plane runs from $-i\infty$ to $i\infty$ with loops, if necessary to ensure that the poles of $\Gamma(b_j - mt), j = 1, 2, \ldots, \alpha$ lie to the right and the poles of $\Gamma(m - a_j + mt), j = 1, 2, \ldots, \beta$ and $\Gamma(1+\beta_j+s+nt), j = 1, 2, \ldots h$ are to the left of the contour.

In the proof is involved the change of order of integration, to justify the same let us first investigate the convergence of the integral (11) in the s-plane. For that we put $s = iy$ (t is considered to be constant) and take the limit as $y \to \infty$ then since

$$\lim_{y \to \infty} \left| \Gamma(x+iy) \right| = (2\pi)^{\frac{1}{2}} \, |y|^{x-\frac{1}{2}} \, e^{-\frac{1}{2}\pi |y|}$$

The absolute value of the above integrand is comparable with

$$\exp\left[-\tfrac{1}{2}\pi \,(2h+2A+2B-D-C-q-r)\, |y| \right] e^{\phi |y|}$$

$$\times \, yR\left[\Sigma f_j - \Sigma l_j + \Sigma \beta_j - \Sigma \alpha_j + \tfrac{1}{2} C - \tfrac{1}{2} D - \tfrac{1}{2} q + \tfrac{1}{2} r\right]$$

where $$\left(\frac{a}{b}\right) = Re^{i\phi}$$

Hence if $2h + 2A + 2B > D + C + q + r$, then the integral is convergent,

either if $$\left|\arg \frac{a}{b}\right| < \left(h+A+B-\tfrac{1}{2} D - \tfrac{1}{2} C - \tfrac{1}{2} q - \tfrac{1}{2} r\right)$$

or if $$\left|\arg \frac{a}{b}\right| = \left(h+A+B-\tfrac{1}{2} D - \tfrac{1}{2} C - \tfrac{1}{2} q - \tfrac{1}{2} r\right)$$

then $$R\left\{\Sigma f_j - \Sigma e_j + \Sigma \beta_j - \Sigma \alpha_j + \tfrac{1}{2} C - \tfrac{1}{2} D - \tfrac{1}{2} q + \tfrac{1}{2} r\right\} < -1.$$

while if $2h + 2A + 2B = D + C + q + r$, then the integral is convergent only

if $$\left|\arg \frac{a}{b}\right| = 0 = \left(h+A+B-\tfrac{1}{2} D - \tfrac{1}{2} C - \tfrac{1}{2} q - r\right)\pi \text{ and}$$

$$R\left\{\Sigma f_j - \Sigma e_j + \Sigma \beta_j - \Sigma \alpha_j + \tfrac{1}{2} C - \tfrac{1}{2} D - \tfrac{1}{2} q + \tfrac{1}{2} r\right\} < -1.$$

The discussion in the t-plane, while considering s to be constant is similar to that given above so we give below only the results.

If $2m\alpha + 2m\beta + 2nh > m\delta + nq + nr + m\gamma$, then the integral is convergent,

either if $$\left|\arg \frac{c^m}{b^n}\right| < \left(m\alpha + m\beta + nh - \tfrac{1}{2} m\delta - \tfrac{1}{2} nq - \tfrac{1}{2} nr - \tfrac{1}{2} m\gamma\right)\pi$$

or if $$\left|\frac{c^m}{b^n}\right| = \left(m\alpha + m\beta + nh - \tfrac{1}{2} m\delta - \tfrac{1}{2} nq - \tfrac{1}{2} nr - \tfrac{1}{2} m\gamma\right)\pi$$

then $$R\left\{\Sigma b_j - \Sigma a_j + \Sigma \beta_j - \Sigma \alpha_j - \alpha - \beta + \tfrac{1}{2}\gamma + \tfrac{1}{2}\delta - \tfrac{1}{2} q + \tfrac{1}{2} r + m\alpha + m\beta - m\delta\right\} < -1.$$

While if $2m\alpha+2m\beta+2nh = m\delta+nq+nr+m\gamma$ then the integral is convergent only if

$$\left|\arg \frac{c^m}{b^n}\right| = 0 = (m\alpha+m\beta+nh-\tfrac{1}{2}m\delta-\tfrac{1}{2}nq-\tfrac{1}{2}nr-\tfrac{1}{2}m\gamma)\pi \text{ and}$$

$$R\{\Sigma b_j-\Sigma\alpha_j+\Sigma\beta_j-\Sigma a_j-\alpha-\beta+\tfrac{1}{2}\gamma+\tfrac{1}{2}\delta-\tfrac{1}{2}q+\tfrac{1}{2}r+m\alpha+m\beta-m\delta\} < -1.$$

Thus we see that the change of order of integration is admissible for the conditions given with (1), as with them integral (2) is absolutely convergent, the inner integral after change of order of integration is also absolutely convegent and the resulting integral (11) is also convergent.

4. If we take $n = m = 1$ in (1), it takes the form

$$\int_0^\infty G\,{}^{A,\,B}_{C,\,D}\left(ax\,\Big|\,{}^{e_1,\,\ldots,\,e_C}_{f_1,\,\ldots,\,f_D}\right) G\,{}^{h,\,o}_{q,\,r}\left(bx\,\Big|\,{}^{\alpha_1,\,\ldots,\,\alpha_q}_{\beta_1,\,\ldots,\,\beta_r}\right) G\,{}^{\alpha,\,\beta}_{\gamma,\,\delta}\left(cx\,\Big|\,{}^{a_1,\,\ldots,\,a_\gamma}_{b_1,\,\ldots,\,b_\delta}\right)dx$$

$$= \frac{1}{b}\,S\left[\begin{array}{l|l|c} \left[\begin{array}{c} h,\ o \\ r-h,\ q \end{array}\right] & \beta_1+1,\ \ldots,\ \beta_r+1\ ;\ \alpha_1+1,\ \ldots\ ;\ \ldots,\ \alpha_q+1 & \\ & & \frac{a}{b} \\ \left[\begin{array}{c} B,\ A \\ C-B,\ D-A \end{array}\right] & e_1,\ \ldots,\ e_C\ ;\ f_1,\ \ldots,\ f_D & \\ & & \frac{c}{b} \\ \left[\begin{array}{c} \beta,\ \alpha \\ \gamma-\beta,\ \delta-\alpha \end{array}\right] & a_1,\ \ldots,\ a_\gamma\ ;\ b_1,\ \ldots,\ b_\delta & \end{array}\right] \qquad (12)$$

When the set of conditions (A) alongwith the following conditions are satisfied.

$$\left|\arg\frac{a}{b}\right| < (h+B+A-\tfrac{1}{2}q-\tfrac{1}{2}D-\tfrac{1}{2}r-\tfrac{1}{2}C)\pi,\ 2(h+B+A) > q+D+r+C,$$

$$\left|\arg\frac{c}{b}\right| < (h+\beta+\alpha-\tfrac{1}{2}q-\tfrac{1}{2}\delta-\tfrac{1}{2}r-\tfrac{1}{2}\gamma)\pi,\ 2(h+\beta+\alpha) > q+\delta+r+\gamma,$$

$$R(\min f_j+\min\beta_j+\min b_j+1) > 0 > R(\max e_j+\max a_j-2)$$

Example I: Putting $h = 1$, $q = o$, $r = 1$, $\beta_1 = 0$, $A = 1$, $B = C$, $\alpha = 1$, $\beta = \gamma$ and specialising the parameters, we get

$$\int_0^\infty e^{-bx}\,x^{\lambda-1}\,{}_cF_{D-1}\left(\begin{array}{c} a'_1,\ \ldots,\ a'_c \\ b'_1,\ \ldots,\ b'_{D-1} \end{array};\ -ax\right)\ {}_\gamma F_{\delta-1}\left(\begin{array}{c} a_1,\ \ldots,\ a_\gamma \\ b_1,\ \ldots,\ b_{\delta-1} \end{array};\ -px\right)dx$$

$$= \frac{\Gamma\lambda}{b^\lambda}\,F\left[\begin{array}{c|l|c} 1 & \lambda & \\ c & a'_1,\ \ldots\ldots,\ a'_c & -\frac{a}{b} \\ \gamma & a_1,\ \ldots\ldots,\ a_\gamma & \\ O & & \\ D-1 & b'_1,\ \ldots\ldots,\ b'_{D-1} & -\frac{p}{b} \\ \delta-1 & b_1,\ \ldots\ldots,\ b_{\delta-1} & \end{array}\right]$$

for $\qquad R(\lambda) > 0,\quad R(b) > 0 \qquad$ (13)

The above result can also be obtained from a result given by Slater [(9) page 54.]

Example II: Assigning special values to the parameters in (12) we have

$$I = \int_0^\infty e^{-(\alpha+b+\beta)x}\ I_\mu(\alpha x)\ I_\nu(\beta x)\ dx$$

$$= \frac{\alpha^{\mu}\,\beta^{\nu}\,\Gamma(1+\mu+\nu)\,2^{-\mu-\nu}}{b^{1+\mu+\nu}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}\,F_2\left[1+\mu+\nu,\ \tfrac{1}{2}+\mu,\ \tfrac{1}{2}+\nu,\ 1+2\mu,\ 1+2\nu\,;\ -\frac{2\alpha}{b},\ -\frac{2\beta}{b}\right]$$

for
$$R(\alpha\pm b\pm\beta) > 0,\ R(\mu+\nu+1) > 0 \tag{14}$$

Also from a result due to Bailey[4]

$$I = \frac{\alpha^{\mu}\,\beta^{\nu}\,2^{-\mu-\nu}\,\Gamma(1+\mu+\nu)}{(\alpha+\beta+b)^{1+\mu+\nu}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}\,F_4\Big[\tfrac{1}{2}(1+\mu+\nu),\ \tfrac{1}{2}(2+\mu+\nu),\ \mu+1,\ \nu+1\,;$$
$$\times\ \frac{\alpha^2}{(\alpha+\beta+b)^2},\ \frac{\beta^2}{(\alpha+\beta+b)^2}\Big] \tag{15}$$

equivalence of both gives

$$F_2[2\alpha,\ \beta,\ \beta',\ 2\beta,\ 2\beta'\,;\ x,y]$$
$$= (1-\tfrac{1}{2}x-\tfrac{1}{2}y)^{-2\alpha}\,F_4\left[\alpha,\ \alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2},\ \beta'+\tfrac{1}{2};\ \frac{x^2}{(2-x-y)^2}\,;\ \frac{y^2}{(2-x-y)^2}\right] \tag{16}$$

Which holds inside the common domain of the series involved. This formula is essentially a generalisation of the well known quadratic transformation [4] page 111 eq. (4)]

$${}_2F_1\{a,\ b\ ;\ 2\,b\ ;\ x\} = (1-\tfrac{1}{2}x)^{-a}\ {}_2F_1\left\{\tfrac{1}{2}\,a,\ \tfrac{1}{2}\,a+\tfrac{1}{2}\ ;\ b+\tfrac{1}{2}\ ;\ \left(\frac{x}{2-x}\right)^2\right\}$$

5. If we put x^2 for x in (12) and make use of the formula (3) weget

$$\int_0^\infty G\,{}^{A,\,B}_{C,\,D}\left(a\,x^2\ \middle|\ {}^{e_1,\,\ldots,\,e_C}_{f_1,\,\ldots,\,f_D}\right) G\,{}^{h,\,o}_{q,\,r}\left(b\,x^2\ \middle|\ {}^{\alpha_1,\,\ldots,\,\alpha_q}_{\beta_1,\,\ldots,\,\beta_r}\right) G\,{}^{\alpha,\,\beta}_{\gamma,\,\delta}\left(cx^2\ \middle|\ {}^{a_1,\,\ldots,\,a_\gamma}_{b_1,\,\ldots,\,b_\delta}\right) dx$$

$$= \frac{1}{2\sqrt{b}}\ S\left[\begin{array}{c|c|c} \begin{bmatrix} h,\ o \\ r-h,\ q \end{bmatrix} & \beta_1+\tfrac{1}{2},\ \ldots,\ \beta_r+\tfrac{1}{2}\ ;\ \alpha_1+\tfrac{1}{2},\ \ldots,\ \alpha_q+\tfrac{1}{2} & \\ \begin{bmatrix} B,\ A \\ C-B,\ D-A \end{bmatrix} & e_1,\ \ldots\ldots,\ e_C\ \ ;\ f_1,\ \ldots\ldots,\ f_D & \frac{a}{b} \\ \begin{bmatrix} \beta,\ \alpha \\ \gamma-\beta,\ \delta-\alpha \end{bmatrix} & a_1,\ \ldots\ldots,\ a_\gamma\ \ ;\ b_1,\ \ldots\ldots,\ b_\delta & \frac{c}{b} \end{array}\right] \tag{17}$$

the conditions of validity being the same as given with (12).

Example I: If we specialize the parameters so as to apply the formula (4) in the G-Functions of (17) we get,

$$\int_0^\infty x^{\lambda-1}\,K_\nu(ax)\,K_\mu(ax)\,K_{\nu'}(bx)\,K_{\mu'}(bx)\,K_{\nu''}(cx)\,K_{\mu''}(cx)\,dx$$

$$= \frac{\pi^{3/2}}{16b}\ S\left[\begin{array}{c|l|c} \begin{bmatrix} 4,\ 0 \\ 0,\ 2 \end{bmatrix} & \tfrac{1}{2}(\nu'+\mu'+1),\ \tfrac{1}{2}(\nu'-\mu'+1),\ \tfrac{1}{2}(\mu-\nu'+1),\ \tfrac{1}{2}(1-\nu'-\mu');\ \tfrac{1}{2},\ 1 & \\ \begin{bmatrix} 0,\ 4 \\ 2,\ 0 \end{bmatrix} & \tfrac{1}{2}\lambda-\tfrac{1}{2},\ \tfrac{1}{2}\lambda\ ;\ \tfrac{1}{2}(\nu+\lambda-1+\mu),\ \tfrac{1}{2}(-\nu+\lambda-1+\mu), & \frac{a^2}{b^2} \\ & \qquad\qquad \tfrac{1}{2}(-\nu+\lambda-1-\mu),\ \tfrac{1}{2}(\nu+\lambda-1-\mu) & \\ \begin{bmatrix} 0,\ 4 \\ 2,\ 0 \end{bmatrix} & 0,\ \tfrac{1}{2}\qquad ;\ \tfrac{1}{2}(\nu''+\mu''),\ \tfrac{1}{2}(\nu''-\mu''),\ \tfrac{1}{2}(\mu''-\nu''),\ \tfrac{1}{2}(-\mu''-\nu'') & \frac{c^2}{b^2} \end{array}\right]$$

$$R(\lambda\pm\nu\pm\mu\pm\nu'\pm\mu'\pm\nu''\pm\mu'') > 0 \tag{18}$$

Next we remark that if the parameters are suitably chosen, an application of the formula will express the integrals of the following types also in terms of the S–function which by (10) can be expressed as hypergeometric series of two variables under appropriate conditions of convergence.

$$(i) \int_0^\infty x^{\lambda-1} J_{\nu}(ax)\, Y_{\nu}(ax)\, J_{\mu}(bx)\, Y_{\mu}(bx)\, J_{\nu}(cx)\, Y_{\rho}(cx)\, dx$$

$$(ii) \int_0^\infty x^{\lambda-1} I_{\mu}(ax)\, K_{\nu}(ax)\, I_{\mu'}(bx)\, K_{\nu'}(bx)\, I_{\mu''}(cx)\, K_{\nu''}(cx)\, dx$$

$$(iii) \int_0^\infty x^{\lambda-1} J_{\mu}(ax)\, J_{\nu}(ax)\, J_{\mu'}(bx)\, J_{\nu'}(bx)\, J_{\mu''}(cx)\, J_{\nu''}(cx)\, dx$$

$$(iv) \int_0^\infty x^{\lambda-1} H^{(1)}_{\nu}(ax)\, H^{(2)}_{\nu}(ax)\, H^{(1)}_{\mu}(bx)\, H^{(2)}_{\mu}(bx)\, H^{(1)}_{\rho}(cx)\, H^{(2)}_{\rho}(cx)\, dx$$

$$(v) \int_0^\infty x^{\lambda-1} W_{k,m}(ax)\, W_{-k,m}(ax)\, W_{\lambda,\mu}(bx)\, W_{-\lambda,\mu}(bx)\, W_{\rho,\nu}(cx)\, W_{-\rho,\nu}(cx)\, dx$$

Particular cases : (*i*) If $A=1$, $B=0$, $C=0$, $D=2$, $h=2$, $q=0$, $r=2$, $\alpha=1$, $\beta=0$, $\gamma=0$, $\delta=2$, $f_1=\frac{1}{4}+\frac{\mu}{2}$, $f_2=\frac{1}{4}-\frac{\mu}{2}$, $\beta_1=\frac{1}{4}+\frac{\nu}{2}$, $\beta_2=\frac{1}{4}-\frac{\nu}{2}$, $b_1=\frac{1}{4}+\frac{\rho}{2}$, $b_2=\frac{1}{4}-\frac{\rho}{2}$ in (17) ; and expanding the S-function into a double series of F_4 by (9) and (10) we get a result due to Bailey[1]

(*ii*) Again by taking $A=1$. $B=0$, $C=0$, $D=2$, $h=2$, $t=2$, $q=0$, $\alpha=2$, $\beta=0$, $\gamma=0$, $\delta=2$, $f_1=\frac{1}{4}+\frac{\rho}{2}$, $f_2=\frac{1}{4}-\frac{\rho}{2}$, $\beta_1=\frac{1}{4}+\frac{\nu}{2}$, $\beta_2=\frac{1}{4}-\frac{\nu}{2}$, $b_1=\frac{1}{4}+\frac{\mu}{2}$, $b_2=\frac{1}{4}-\frac{\mu}{2}$ in (17); and using the relations (9) and (10), we get known result due to Sharma[8]

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A finite integral Involving H-function

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Abstract

The integral

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1} H_{p\ q}^{m\ n}\left[zx^k(t-x)^s \,\middle|\, \begin{matrix} a_r, e_r) \\ b_s, f_s) \end{matrix}\right] dx \tag{A}$$

has been evaluated and its particular cases are derived.

1. Introduction

MacRobert[4] evaluated an integral

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} E\left[\ p\,;\, a_r : q\,;\, \beta_s : z\, x^l\,(1-x)^m\ \right] dx \tag{1·1}$$

where l and m have same sign. Recently Sharma[6] generalized it as

$$\int_0^a x^{\beta-1}(x+y)^{-a-\beta} G_{r\ s}^{\lambda\ \mu}\left[z\, x^{l-m}(x+y)^m \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right] dx \tag{1·2}$$

for positive integral values of l and m.

In this paper an attempt is made to generalize it further in the form (A) in which the parameters are subject to the conditions that

(*i*) $R(\alpha) > 0$, $R(\beta) > 0$;

(*ii*) k and s are to be non-negative integers not both zero ;

(*iii*) $0 \leqslant m \leqslant q$; $0 \leqslant n \leqslant p$; e's and f's are all positive ;

(*iv*) $\lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ or $\lambda \geqslant 0$, $|\arg z| < \frac{1}{2}\lambda\pi$, $R(\mu+1) < 0$ where λ and μ have the values

$$\sum_{j=1}^{n}(e_j) - \sum_{j=n+1}^{p}(e_j) + \sum_{j=1}^{m}(f_j) - \sum_{j=m+1}^{q}(f_j) \text{ and } \tfrac{1}{2}(p-q) + \sum_{j=1}^{q}(b_j) - \sum_{j=1}^{p}(a_j)$$

respectively.

Proof: We evaluate (A) by term by term integration. Let $x = tv$. Then it becomes

$$t^{\alpha+\beta-1}\int_0^1 v^{\alpha-1}(1-v)^{\beta-1} H_{p\ q}^{m\ n}\left[z\, t^{k+s}\, v^k\,(1-v)^s \,\middle|\, \begin{matrix} (a_r, e_r) \\ (b_s, f_s) \end{matrix}\right] dv \tag{1·3}$$

Substituting the value of H-function given by Fox[2] in the above we have

$$t^{\alpha+\beta-1}\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\left[\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+e_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-e_j\xi)}\, z^{\xi}\, t^{\xi(k+s)}\, v^{k\xi}(1-v)^{s\xi}\, d\xi\right]dv \tag{1·4}$$

Now changing the order of integration we get

$$t^{\alpha+\beta-1}\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+e_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-e_j\xi)}\, z^{\xi}\, t^{\xi(k+s)} \left\{\int_0^1 v^{k\xi+\alpha-1}(1-v)^{s\xi+\beta-1}\,dv\right\} d\xi \tag{1·5}$$

so that on evaluating the inner v–integral by

$$B(p',q') = \int_0^1 y^{p'-1}(1-y)^{q'-1}\,dy,\quad R(p')>0,\ R(q')>0 \tag{1·6}$$

precisely the result can be expressed as follows

$$t^{\alpha+\beta-1} H^{m,\ n+2}_{p+2,\ q+1}\left[z\,t^{k+s}\,\middle|\,\begin{matrix}(1-\alpha,k),(1-\beta,s),(a_r,e_r)\\ (b_s,f_s),(1-\alpha-\beta,k+s)\end{matrix}\right] \tag{1·7}$$

The change of the order of integration in (1·4) is justified as we observe that the inner v–integral is absolutely convergent if $R(\alpha+k)>0$, $R(\beta+s)>0$ and the ξ-integral converges in atleast one of cases referred in (iv).

Particular cases:

(*a*) If $s=0$ and $k=\sigma$, it is simple to modify the steps in the derivation of (A). It comes out to be

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1} H^{m,\ n}_{p,\ q}\left[z x^{\sigma}\,\middle|\,\begin{matrix}(a_r,e_r)\\ (b_s,f_s)\end{matrix}\right]dx = \Gamma\beta\, t^{\alpha+\beta-1}\, H^{m,\ n+1}_{p+1,\ q+1}\left[z t^{\sigma}\,\middle|\,\begin{matrix}(1-\alpha,\sigma),(a_r,e_r)\\ (b_s,f_s),(1-\alpha-\beta,\sigma)\end{matrix}\right] \tag{1·8}$$

(*b*) If we put σ, all e's and f's equal to unity in (*a*) the result (1, p. 417) is obtained.

(*c*) If all e's and f's are equal to unity, $m=p$, $n=1$, $p=q+1$, $q=p$, $a_1=1$, then by (1, p. 444)

$$G^{p,\ 1}_{q+1,\ p}\left[x\,\middle|\,\begin{matrix}1,\eta_q\\ \xi_p\end{matrix}\right] = E[\,p\,;\xi_r : q\,;\eta_s : x\,] \tag{1·9}$$

we obtain the value of (1·1) as given by MacRobert for positive integral values of k and s.

(*d*) If $m = 1$, $q = 2$, $p = n = 0$, $b_1 = 0$, $f_1 = 1$, $b_2 = -\nu$, $f_2 = \mu$ in (*a*) then by virtue of

$$J_{\nu}^{\mu}(x) = H_{0\ 1}^{1\ 0}\left[x \,\middle|\, (0, 1), (-\nu, \mu)\right] \tag{1·10}$$

we obtain

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1} J_{\nu}^{\mu}(z x^{\sigma})\, dx = \Gamma\beta\, t^{\alpha+\beta-1}$$

$$H_{1\ 3}^{1\ 1}\left[zt^{\tau} \,\middle|\, \begin{matrix} (1-\alpha, \sigma) \\ (0, 1), (1-\alpha-\beta, \sigma), (-\nu, \mu) \end{matrix}\right] \tag{1·11}$$

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Recovery of Fertilizer Nitrogen by Oats as affected by Nitrogen Fertilization Levels and Soil Moisture Supply[1]

By

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Abstract

In a pot culture experiment with oats the recovery of fertilizer nitrogen as affected by the levels of nitrogen fertilization (*i.e.*, 150 and 300 mg nitrogen as nitrate per Kg. soil) and moisture regime (*i.e.*, 'Dry' and 'Moist' regime, corresponding respectively to giving irrigation when 80% and 50% of the available soil moisture were exhausted) was studied using fertilizer nitrogen labelled with ^{15}N. The recovery, ranging from 25·7% to 55·6%, was decreased due to increase in level of nitrogen fertilization and due to drier moisture regime. As compared to the isotopic method, the difference method of determining nutrient utilization yielded lower values which ranged from 8·2% to 50·2%, depending on the level of nitrogen fertilization and moisture supply. The underestimation of the recovery values was particularly greater in magnitude at higher levels of nitrogen fertilization and 'drier' moisture regime.

Introduction

The utilization of a nutrient from a fertilizer source has often been determined by subtracting the amount of that nutrient in unfertilized plants from that in the fertilized ones. This, the 'difference method', has drawbacks that on one hand the plants grown under the condition of deficient nutrient supply are compared with those amply supplied with that nutrient and on the other hand the effect of the applied fertilizer on the availability of that nutrient already present in soil is not taken care of. Use of a labelled source, however—be the label a radioactive or a stable isotope—offers the possibility that through the application of the principle of isotopic dilution the simultaneous contribution of two sources of a nutrient be determined without above limitations.

Utilization of the fertilizer nitrogen by plants has been investigated by using the stable isotope ^{15}N. Most of such studies have shown that the recovery of fertilizer nitrogen by the main crop is rather low (Bartholomew *et al.* 1950, MacVicar *et al.* 1950, Turtschin *et al.* 1960, Viets, 1960). This paper reports the results of an experiment which was conducted to study the utilization of fertilizer nitrogen by oats (*Avena sativa*) as affected by the nitrogen fertilization levels and the soil moisture supply. The experiment was also aimed at studying how the difference method would compare with the isotopic method in determining the nitrogen recovery under conditions of the above variables.

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Experimental

The experiment consisted of culture of oats similar to pot-culture in 2 liter beakers having a mixture of 0·75 Kg. fine quartz-sand and 1 Kg. of a loam soil (Northern Blackforests, W. Germany, containing 0·239% total N, 0·023% lactate soluble P_2O_5, and 0·026% lactate soluble K_2O ; pH value of 6·0 and moisture content of 22·7% at 0·3 atm. and 9·8% at 15 atm. tensions on the dry weight basis). In a factorial design, all combinations of the three levels of nitrogen, N_0, N_{150} ; N_{300} (*i.e.*, 0, 150 and 300 mg nitrogen per pot), and two moisture regimes, 'moist' and 'dry', (corresponding to irrigation when respectively 50% and 80% of the available water where exhausted) were tested. Each pot received an uniform supply of 200 mg. P_2O_5 (as finely powdered $CaHPO_4$ mixed with dried soil) and 250 mg. K_2O (as K_2SO_4 in no nitrogen pots and as KNO_3 in the rest). The balance of nitrogen, as necessitated by the treatments, was supplied through $Ca(NO_3)_2.4H_2O$. To lable the fertilizer nitrogen, 20% of the total amount was supplied through a KNO_3 salt containing 95·7 per cent excess ^{15}N. The salts supplying nitrogen and potassium were mixed with the soil as solutions before plantin g. The pots, initially planted with 15 seeds of oats (var. Carsten's Phönix), finally carried 12 plants each. The experiment was terminated 70 days after sowing (*i.e.* 45 days after the onset of moisture regime treatments) by harvesting the whole shoots, which were immediately dissected into leaf, stem and ear, freeze-dried, weighed and prepared for analysis. The content of total nitrogen in the dried plant material was determined by Micro-Kjeldahl procedure along the lines described by Peach and Tracey (1956). For ^{15}N-assay, from the arc-emission-spectrograph of N_2 gas, the frequency of ^{15}N in the total nitrogen was calculated by comparing the optical densities of $^{28}N_2$ and $^{29}N_2$ lines (corresponding to 3159 Å and 3162 Å, respectively) as described by Schumacher (1965). Per cent recovery of fertilizer nitrogen by isotopic method (R_i) and by difference method (R_d) were calculated as follows :

$$R_i = \frac{\text{frequency of } ^{15}N \text{ in plant nitrogen}}{\text{frequency of } ^{15}N \text{ in fertilizer nitrogen}} \times 100 \text{ and}$$

$$R_d = \frac{\begin{bmatrix}\text{nitrogen-uptake by} \\ \text{fertilized plants}\end{bmatrix} - \begin{bmatrix}\text{nitrogen-uptake by} \\ \text{unfertilized plants}\end{bmatrix}}{\text{nitrogen added to the soil as fertilizer.}} \times 100$$

Results and Discussion

The main effects of the levels of nitrogen and moisture regimes as well as their combinations on the dry matter yield, total nitrogen uptake and utilization of fertilizer nitrogen have been summarised in the following Table.

The mean effect of nitrogen fertilization on the dry matter yield was not significant. The yield was markedly low at 'dry' regime as compared to 'moist' one. Increasing rates of nitrogen fertilization resulted in decreasing yield increment at 'moist' regime whereas at 'dry' moisture regime nitrogen fertilization caused reduction in yield as compared to no nitrogen control.

Total nitrogen uptake increased as the nitrogen supply was increased, the rise due to the first dose (*i.e.*, N_{150}) being higher than due to the second one (*i.e.*, N_{300}). At 'dry' regime the total uptake was considerably lesser than at the 'moist' ; however, this effect of soil moisture regimes was not as marked as their effect on the dry matter production.

Whereas the absolute amount of fertilizer nitrogen recovered by plants increased as the fertilization rate was raised, the per cent recovery of applied nitrogen decreased. The recovery was conspicuously low at 'dry' moisture regime. A comparison of recovery data as worked out by the isotopic method with those determined by the difference method reveals that the latter method underestimated the recovery values. The magnitude of this underestimation was greater at higher fertilizer dose in comparison to the lower one and at 'dry' regime as compared to 'moist' regime.

TABLE 1

Effect of levels of nitrogen and mositure on the dry matter production, total nitrogen removal and recovery of fertilizer nitrogen by oat shoots

Treatments	Dry matter yield g/pot	Total N removed mg/pot	Per cent N derived from fertilizer (isotopic method)	N (mg/pot) removed from the fertilizer as estimated by		% recovery of fertilizer nitrogen as estimated by		b/a
				Isotopic method	Difference method	Isotopic method a	Diff. rence method b	
Levels of N (average over moisture regimes) :								
No nitrogen	9·4	145·6	–	–	–	–	–	–
150 mgN/pot	9·7	193·1	35·3	68·7	47·6	45·8	31·7	0·69
300 mgN/pot	9·7	209·1	48·0	101·0	63·5	33·7	21·1	0·62
L. S. D. 5%	**	7·1	1·3	6·7	8·8	3·1	3·8	–
Moisture regimes (average over levels of nitrogen) :								
Moist	11·9	209·6	43·2	104·2	88·8	48·6	42·1	0·86
Dry	7·3	155·5	40·1	65·5	22·2	30·8	10·7	0·34
L. S. D. 5%	0·5	6·6	1·3	6·7	8·8	8·1	3·8	–
Moisture regime × nitrogen levels :								
Moist, no nitrogen	11·2	150·4	–	–	–	–	–	–
Moist, 150 mgN/pot	12·2	225·7	36·9	83·4	75·3	55·6	50·2	0·91
Moist, 300 mgN/pot	12·3	252·7	49·4	124·9	102·3	41·6	34·1	0·82
Dry, no nitrogen	7·5	140·8	–	–	–	–	–	–
Dry, 150 mgN/pot	7·2	160·6	33·6	53·9	19·8	35·9	13·2	0·37
Dry, 300 mgN/pot	7·1	165·8	46·6	77·1	24·6	25·7	8·2	0·32
L. S. D. 5%	9·0	11·5	**	9·4	12·4	**	5·4	–

**Not significant

A lack of substantial growth response of plants to nitrogen fertilization could be attributed to the presence of appropriate amount of nitrogen in the soil. Due to the luxury consumption of available nitrogen, the amount of total nitrogen taken up by plants increased as the level of nitrogen fertilization was raised in spite of little increase in the dry matter production. The percentage of applied nitrogen recovered by plants decreased as the level of nitrogen fertilization was raised. In all probability, the limiting effect of some other growth factors became accentuated at the higher level of nitrogen fertilization so that increased amount of available nitrogen could not be utilized by the plants in proportion to its supply. Very low recovery values for the 'dry' moisture regime were primarily due to highly restricted growth resulting from the insufficient moisture supply for plants at this regime.

The two methods of calculating recovery of fertilizer nitrogen gave velues which differed considerably from each other. Although the nitrogen utilization values as worked out by isotopic method could also be lower than the actual ones because of microbial immobilisation of fertilizer nitrogen and accelerated mineralisation of soil nitrogen upon the addition of fertilizer in the soil, these values are more reliable than those yielded by the difference method; after all 'the label in the plant would come from the applied fertilizer only (Michael and Machold 1957). The isotopic method can, therefore, be taken as a standard for comparison. The most conspicuous fact that emerged out of this comparison is that the difference method very much underestimated the values for fertilizer utilization particularly under the conditions of lower moisture supply. Under 'dry' moisture regime the recovery values for fertilizer nitrogen estimated by difference method were almost one third of that by isotopic method. This discrepancy seems to arise due to the basic assumptions on which the difference method works. Here it is assumed that the amount of soil nitrogen taken up by plants would be same both in the presence as well as in the absence of a nitrogen fertilizer. The validity of this assumption is naturally limited (Behrens 1955, Kurtz *et al.* 1961, Michael and Machold 1957). This assumption could perhaps hold under such situation, when with increasing rates of nitrogen fertilization a linear increase in the total nitrogen uptake occurred proportionate to the increased total nitrogen supply in the soil. If such an increase were not to occur, the assumption would be invalid because in the presence of fertilizer nitrogen, then, a relatively smaller amount of soil nitrogen would be taken up by plants, as logically the plants should cover their nitrogen needs from soil and fertilizer sources proportionate to the amount of the available nitrogen present in each of these sources. The result would be that while calculating the recovery by difference method a value for soil nitrogen bigger than the actual one would be subtracted from the total nitrogen ond consequently a lower value for recovery of fertilizer nitrogen would be obtained. The results of present experiments could serve as an example supporting this contention. In this study, as the level of nitrogen fertilization was raised, the total nitrogen uptake did not increase linearly but at a diminishing rate and the marginal increments were particularly smaller under the 'dry' regime as compared to the 'moist one'. The values for fertilizer nitrogen utilization as determined by difference method were, therefore, lower in comparison to the isotopic method and the magnitude of the deficit was particularly greater at drier moisture regime and higher levels of nitrogen fertilization. It is probable that under 'dry' moisture regime conditions, as the level of nitrogen fertilization was raised, the root growth was adversely affected, in consequent of which the root system of fertilized plants could not take up as much soil nitrogen as that of the unfertilized control; hence, while calculating nitrogen utilization by difference method a much

bigger amount for soil nitrogen was deducted than the actual one, and the recovery values worked out to be very low.

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Effect of Some Rare-elements on Nitrification by Nitrobacter agilis (In liquid culture medium)

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Introduction

The nutrient requirements of nitrifying bacteria are simple. Some elements namely magnesium and phosphorus[1], iron[2] and molybdenum[3] have been found to be essential for them. Copper[4] and manganese[5], although not essential for their growth, yet have been observed to accelerate the process of nitrification, when they are present in traces. Khare and Tandon[6] have studied the effect of some rare elements on nitrosofication and have found that where the optimum amounts of salts of rare elements are used, the growth of Nitrosomonas is increased, in some cases the growth is twice that of the growth in normal culture medium. According to their observations, the order of the efficiency of trace elements is as follows :

Indium sulphate > Zirconium chloride > Thallous sulphate >

Beryllium sulphate > Osmium tetroxide > Cerous chloride.

In order to understand the effect of some rare elements on nitrate formation, in liquid culture medium by Nitrobacter *agilis*, we have carried out extensive experiments. The results obtained with thallous sulphate, beryllium sulphate, auric chloride, indium sulphate, thorium sulphate, uranyl sulphate and zirconium chloride are recorded here.

Experimental

For the systematic study of the effect of thallous sulphate, indium sulphate, thorium sulphate, beryllium sulphate, auric chloride, uranyl sulphate and zirconium chloride on nitrite oxidation by Nitrobacter *agilis*, the following procedure was adopted.

The following solutions were taken :

Solution A – sterilized sodium nitrite solution containing one milligram of nitrogen per ml of the sulution.

Solution B – containing all the constituents of Fred and Davenport's medium except sodium nitrite.

Solution C – $\frac{M}{10,000}$ solution of the rare salts used.

30 ml of Fred and Davenport's medium were taken for each experiment with different amounts of a particular salt solution and the volume of the whole solution was made upto 50 ml by adding the requisite quantity of distilled water. The composition of the mixture with thallous sulphate is give here. The following nine sets were taken separately in duplicate in 250 ml conical flasks :

(*i*) 30 ml solution B + 20 ml distilled water

(*ii*) 30 ml solution B + 1·0 ml M/10,000 Tl_2SO_4 solution + 19·0 ml distilled water.

(*iii*) 30 ml solution B + 2·0 ml M/10,000 Tl_2SO_4 solution + 18·0 ml distilled water.

(*iv*) 30 ml solution B + 4·0 ml M/10,000 Tl_2SO_4 solution + 16·0 ml distilled water.

(*v*) 30 ml solution B + 6·0 ml M/10,000 Tl_2SO_4 solution + 14·0 ml distilled water.

(*vi*) 30 ml solution B + 8·0 ml M/10,000 Tl_2SO_4 solution + 12·0 ml distilled water.

(*vii*) 30 ml solution B + 10·0 ml M/10,000 Tl_2SO_4 solution + 10·0 ml distilled water.

(*viii*) 30 ml solution B + 15·0 ml M/10,000 Tl_2SO_4 solution + 5·0 ml distilled water.

(*ix*) 30 ml solution B + 20·0 ml M/10,000 Tl_2SO_4 solution

0·2 ml solution A was added to each of the above flasks. All the flasks were sterilized at 15 lbs pressure for 15 minutes in an electric autoclave. After sterilization the flasks were allowed to cool and then 1·0 ml inoculum of a pure culture of Nitrobacter *agilis* was introduced into each of the flasks and the latter were then kept for incubation.

Nitrite was estimated in each flask after 48, 96, 168, 240 ond 360 hours by Griess–Ilosovay method[7]. Nitrite+nitrate nitrogen was also estimated at the beginning and at the end of each experiment by brucine method[8].

Similar experiments were arranged employing each of the following salts in place of thallous sulphate :—beryllium sulphate, thorium sulphate, indium sulphate, auric chloride, uranyl sulphate and zirconium chloride.

TABLE 1

Nitrite oxidation in presence of thallous sulphate (mol. wt. 500·8)

ml of thallous sulphate solution in 50 ml	Nitrite nitrogen left at different intervals of time (in mg)				
	Time in hours				
	48	96	168	240	360
Control	0·156090	0·087120	0·020328	0·005445	
1·0	0·163350	0·092928	0·031218	0·015246	
2·0	0·170610	0·117975	0·047916	0·031218	
4·0	0·177870	0·119790	0·069696	0·031218	
6·0	0·181500	0·125235	0·087846	0·046464	0·004356
8·0	0·185130	0·130680	0·090750	0·047916	0·024684
10·0	0·190575	0·166980	0·148830	0·132495	0·110715
15·0	0·192390	0·170610	0·152460	0·152460	0·152460
20·0	0·192390	0·177870	0·161535	0·161535	0·161535

Control = containing no thallous sulphate

TABLE 2

Nitrite oxidation in presence of indium sulphate (mol. wt. 679·5)

Indium sulphate solution added in ml in 50 ml.	Nitrite nitrogen left at different intervals of time (in mg.) Time in hours				
	48	96	168	240	360
Control	0·156090	0·087120	0·020328	0·005445	–
1·0	0·150645	0·085305	0·011016	–	–
2·0	0·141570	0·077019	0·004356	–	–
4·0	0·136125	0·071874	0·002178	–	–
6·0	0·130680	0·055176	–	–	–
8·0	0·120615	0·049368	–	–	–
10·0	0·116160	0·024684	–	–	–
15·0	0·154275	0·095106	0·031218	0·011616	–
20·0	0·190575	0·174240	0·157905	0·132495	0·112530

Control = containing no indium sulphate

TABLE 3

Nitrite oxidation in presence of thorium sulphate (mol. wt. 424·12)

Thorium sulphate solution added in ml in 50 ml.	Nitrite nitrogen left at different intervals of time (in mg.) Time in hours				
	48	96	168	240	360
Control	0·156090	0·087120	0·020328	0·005445	–
1·0	0·148830	0·078045	0·018876	–	–
2·0	0·132495	0·067518	0·004719	–	–
4·0	0·118338	0·056265	–	–	–
6·0	0·096722	0·042108	–	–	–
8·0	0·133584	0·066066	0·011616	–	–
10·0	0·157905	0·103455	0·059895	0·010527	–
15·0	0·179685	0·148830	0·118338	0·066429	0·010164
20·0	0·192390	0·176055	0·176055	0·176055	0·176055

Control = containing no thorium sulphate

TABLE 4

Nitrite oxiadation in presence of beryllium sulphate (mol. wt. 105·0)

Beryllium sulphate solution added in ml in 50 ml	Nitrite nitrogen left at different intervals of time (in mg.) Time in hours				
	48	96	168	240	360
Control	0·156090	0·087120	0·020328	0·005445	–
1·0	0·150645	0·084216	0·015246	0·003267	–
2·0	0·148830	0·076956	0·013794	0·002904	–
4·0	0·148830	0·075504	0·005808	–	–
6·0	0·145200	0·078408	0·007986	–	–
8·0	0·150645	0·088045	0·016698	0·004719	–
10·0	0·168795	0·089298	0·033396	0·010527	–
15·0	0·176055	0·133584	0·088209	0·040656	0·005808
20·0	0·200000	0·200000	0·200000	0·200000	0·200000

Control = containing no beryllium sulphate

TABLE 5

Nitrite oxidation in presence of auric chloride (mol. wt. 303·7)

Auric chloride solution added in ml in 50 ml	Nitrite nitrogen left at different intervals of time (in mg.) Time in hours				
	48	96	168	240	360
Control	0·156090	0·087120	0·020328	0·005445	–
1·0	0·150645	0·078408	0·015246	0·002178	–
2·0	0·139755	0·067518	0·010164	–	–
4·0	0·148830	0·071148	0·037940	–	–
6·0	0·157904	0·128680	0·080586	0·022506	0·003267
8·0	0·168795	0·141570	0·110715	0·076230	0·040656
10·0	0·186945	0·179685	0·165165	0·145200	0·132495
15·0	0·200000	0·200000	0·200000	0·200000	0·200000
20·0	0·200000	0·200000	0·200000	0·200000	0·200000

Control = containing no auric chloride

TABLE 6

Nitrite oxidation in presence of uranyl sulphate (*mol. wt.* 366·0)

Uranyl sulphate solution added in ml in 50 ml	Nitrite nitrogen left at different intervals of time (in mg.)				
	Time in hours				
	48	96	168	240	360
Control	0·156090	0·087120	0·020328	0·005445	–
1·0	0·124146	0·057354	0·009986	–	–
2·0	0·096195	0·030855	–	–	–
4·0	0·110352	0·044286	–	–	–
6·0	0·125598	0·067518	0·013794	–	–
8·0	0·139755	0·074052	0·022506	0·007623	–
10·0	0·150645	0·089298	0·029766	0·013794	–
15·0	0·176055	0·112530	0·041745	0·015246	–
20·0	0·192390	0·119790	0·056265	0·021054	–

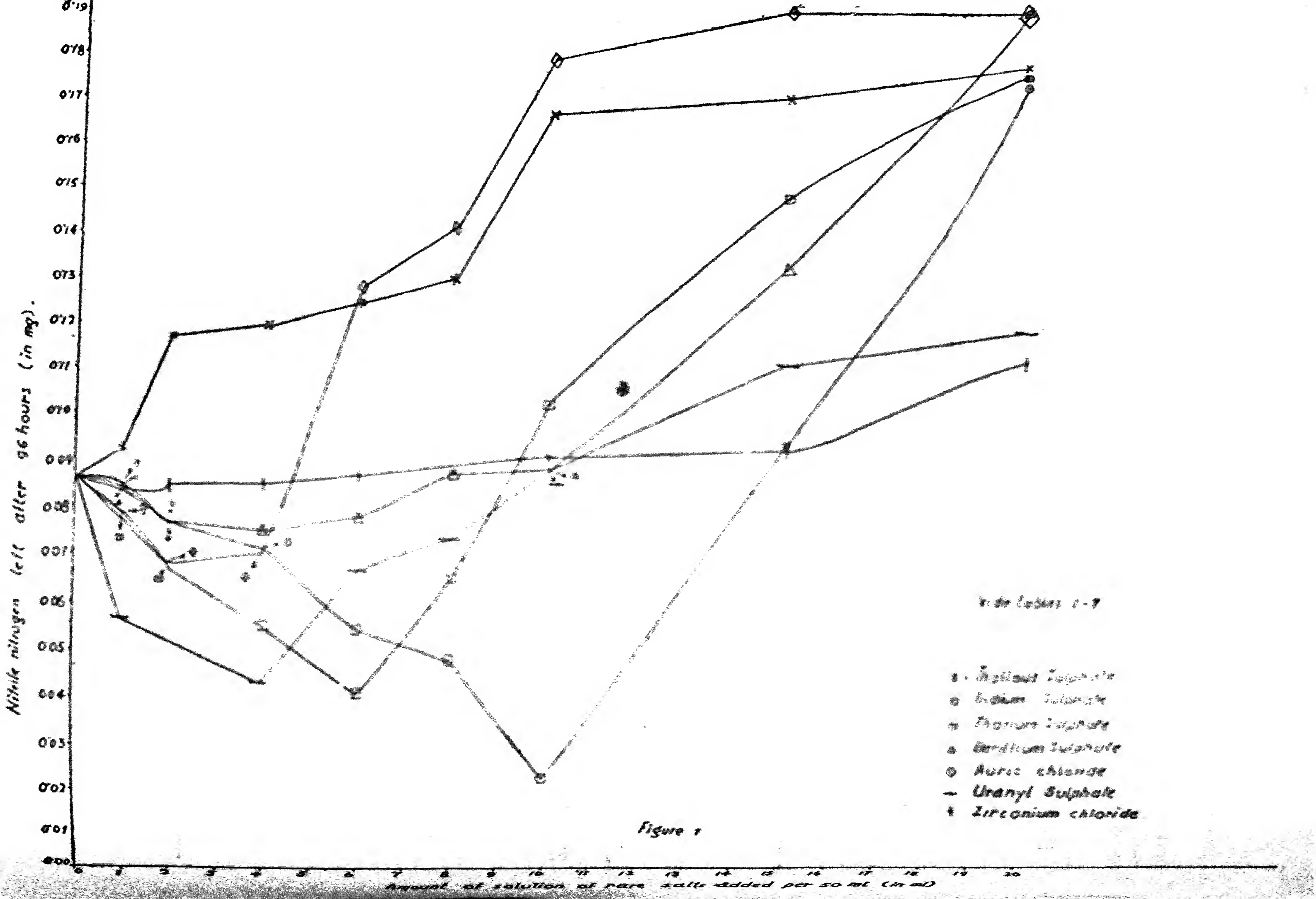
Control = containing no uranyl sulphate

TABLE 7

Nitrite oxidation in presence of zirconium chloride (*mol. wt.* 233·22)

Zirconium chloride solution added in ml in 50 ml	Nitrite nitrogen left at different intervals of time (in mg.)				
	Time in hours				
	48	96	168	240	360
Control	0·156090	0·087120	0·020328	0·005445	–
1·0	0·148830	0·084216	0·016698	0·002904	–
2·0	0·145200	0·084942	0·018513	0·005808	–
4·0	0·141570	0·085668	0·019965	0·004719	–
6·0	0·145200	0·087846	0·018876	0·004356	–
8·0	0·150645	0·086394	0·019239	–	–
10·0	0·152460	0·091476	0·022506	0·005808	–
15·0	0·152460	0·094106	0·021780	0·004709	–
20·0	0·166980	0·113256	0·031218	0·008712	–

Control = containing no zirconium chloride


Nitrite nitrogen left after 96 hours (in mg).
Amount of solution of rare salts added per 50 ml (in ml)
0·00
0·01
0·02
0·03
0·04
0·05
0·06
0·07
0·08
0·09
0·10
0·11
0·12
0·13
0·14
0·15
0·16
0·17
0·18
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11
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13
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17
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19
20
Auric chloride
Uranyl Sulphate
Zirconium chloride

Figure 1

Discussion*

Khare[6] *et al.*, earlier observed that the optimum amount of thallous sulphate increases the activity of Nitrosomonas but it is clear from my results tabulated in Table 1 (Fig. I) that the presence of even the smallest amount of thallous sulphate in the medium decreases the rate of nitrate formation indicating that the activity of Nitrobacter is reduced in the presence of increasing amount of thallous sulphate. The decrease in the rate of nitrate formation in presence of thallous sulphate in the medium is directly proportional to the increase in the amount of thallous sulphate. The rate of nitrite oxidation starts decreasing when only 1 ml of M/10,000 Tl_2SO_4 solution is present in 50 ml of the medium. The rate of nitrite oxidation is very slow when 10 ml of M/10,000 Tl_2SO_4 is present in 50 ml culture medium ; after 360 hours, only less than half of the nitrite taken is oxidised to nitrate, whereas in the medium which has no thallous sulphate, the whole of nitrite is oxidised.

However, contrary to thallous sulphate the presence of beryllium sulphate, thorium sulphate, indium sulphate, auric chloride, uranyl sulphate and zirconium chloride in the medium stimulates the activity of nitrate-forming bacterium (as is clear from Tables 2 to 7, Fig. 1). The rate of nitrite oxidation in presence of these rare salts is accelerated, but after a certain optimum the rate of nitrite oxidation starts decreasing, and finally it stops.

Beryllium sulphate, auric chloride, and zirconium chloride are found to be relatively less effective in stimulating the process of nitrite oxidation as compared to thorium sulphate, indium sulphate and uranyl sulphate. Uranyl sulphate produces a marked increase in the rate of nitrite oxidation even when 2·0 ml of M/10,000 of its solution is present in 50 ml culture medium, but in the case of indium sulphate, for the same increase greater amount of the substance (10 ml of M/10,000 in 50 ml culture medium) is required.

It is also clear from the Tables that Nitrobacter *agilis* has less capacity to tolerate higher concentration of beryllium sulphate and auric chloride than it has for thorium sulphate, indium sulphate, uranyl sulphate and zirconium chloride.

The order of the efficiency of rare-salts in increasing nitrite oxidation by Nitrobacter *agilis* may be put as follows :

Uranyl sulphate > Thorium sulphate > Indium sulphate > Beryllium sulphate > Auric chloride > Zirconium chloride > Thallous sulphate.

Thus, it appears that except thallous sulphate probably all the rare salts employed activate the enzyme system present in the cells as a result of which the nitrite oxidation capacity of bacterium increases, whereas thallous sulphate deactivates the enzyme system and therefore nitrite oxidation capacity of the bacterium decreases.

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*Nitrite + nitrate nitrogen has been estimated at the beginning and at the end of each experiment and it is found that it remains the same. There is no formation of ammonia which thereby shows that there is no loss of nitrogen during the conversion of nitrite to nitrate and that all the nitrite which disappears changes only to nitrate.

Abstract

The effect of thallous sulphate, indium sulphate, thorium sulphate, beryllium sulphate, auric chloride, uranyl sulphate and zirconium chloride on nitrite oxidation by Nitrobacter *agilis* in liquid culture medium, has been studied. The experimental results indicate that except for thallous sulphate, the presence of other salts in the culture medium upto a certain optimum concentration enhances the rate of nitrification. The presence of thallous sulphate on the other hand in the culture medium causes retardation in nitrification by Nitrobacter *agilis*.

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Operational representations and Hypergeometric Functions of three variables

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Abstract

The operational representations of the generalized Appell's functions and other hypergeometric functions of three variables defined by Lauricella[3], Saran[4,5] and Srivastava[7,8,9,10], are given in the present paper.

1. Introduction: Chak[2] has used an operator $\{x^k D\}$ to define a class of polynomials

$$(1\cdot1) \qquad G_{n,k}^{(\alpha)}(x) = x^{-\alpha-kn+n} \; e^{x}(x^k D)^n \{e^{-x} x^{\alpha}\},$$

and Al–Salam[1] has used the operator $\theta=x+x^2D$ to derive the operational relations and some generating functions for known polynomials. Recently the particular case of (1·1) *viz.* $K=2$ has been studied by Singh[6], who has given the following formulae

$$(1\cdot2) \qquad \Omega^n \{x^{\alpha}\} = (\alpha)_n \, x^{\alpha+n},$$

and

$$(1\cdot3) \qquad \left(\frac{1}{\Omega}\right)^n \left\{\frac{1}{x^{\alpha-1}}\right\} = \frac{(-1)^n x^{-\alpha-n+1}}{(\alpha)_n},$$

where $\qquad \Omega = x^2 \dfrac{d}{dx}.$

All the results of Al–Salam[1] can be derived by the operator $\Omega = x^2\, d/dx$ only. Therefore there is no use of adding x in x^2D as Al–Salam[1] has added in his operator θ. The generating functions for known polynomials and other relations, which cannot be found easily by old methods, can be found out very easily by the operator Ω.

In each result given in § 2, the fact is kept in mind that the inverse of the differential operator is essentially an integral one in character. Without applying (1·3), to integrate a function n times, is tedious job. Lauricella[3] in the year 1893 conjectured the existence of ten hypergeometric functions of three variables, in addition to F_A, F_B, F_C and F_D defined and studied by him. These ten functions *viz*, F_E, F_F, F , F_K, F_M, F_N, F_P, F_R, F_S and F_T have been defined by Saran[7].

Again Saran[5] has defined nine more hypergeometric functions of three veriables, *viz.* $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7, \varphi_8$ and φ_9, Srivastava[7,8,9,10] has also defined three new and distinct hypergeometric functions of three arguments *viz*, H_A, H_B, and H_C.

Now in the present paper, we shall give the operational representations of the above defined hypergeometric functions of three veriables, with the help of the relations (1·2) and (1·3).

2. In establishing the results the following notations will be used :

(2·1) $$\Omega = x^2\frac{d}{dx}, \; \omega = y^2\frac{d}{dy}, \; \phi = z^2\frac{d}{dz}, \text{ and } \theta = \eta^2\frac{d}{d\eta}.$$

Now consider

$$\left(1+\frac{\phi}{\Omega}\right)^{-\alpha_1}\left(1+\frac{\omega}{\Omega}\right)^{-\beta_2}\left\{\frac{y^{\alpha_2}\, z^{\beta_1}}{x^{\gamma_1-1}}\, {}_2F_1\left(\alpha_2, \beta_1 ; \gamma_1 ; \frac{yz}{x}\right)\right\}$$

which can be expanded in the form of

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}(\alpha_1)_m\left(\frac{\phi}{\Omega}\right)^m \sum_{n=0}^{\infty}\frac{(-1)^n(\beta_2)_n}{n!}\left(\frac{\omega}{\Omega}\right)^n \sum_{p=0}^{\infty}\frac{(\alpha_2)_p\,(\beta_1)_p}{(\gamma_1)_p\, p!}\frac{y^{\alpha_2+p}\, z^{\beta_1+p}}{x^{\gamma_1-1+p}}$$

or

$$\sum_{m,n,p=0}^{\infty}\frac{(-1)^{m+n}(\alpha_1)_m(\alpha_2)_p}{m!\,n!\,p!}\frac{(\beta_1)_p(\beta_2)_n}{(\gamma_1)_p}\frac{\phi^m\,\omega^n}{\Omega^{m+n}}\left\{\frac{y^{\alpha_2+p}\, z^{\beta_1+p}}{x^{\gamma_1-1+p}}\right\}$$

Applying (1·2) and (1·3), we have

$$\left(1+\frac{\phi}{\Omega}\right)^{-\alpha_1}\left(1+\frac{\omega}{\Omega}\right)^{-\beta_2}\left\{\frac{y^{\alpha_2}\, z^{\beta_1}}{x^{\gamma_1-1}}\, {}_2F_1\left(\alpha_2, \beta_1 ; \gamma_1 ; \frac{yz}{x}\right)\right\}$$

$$=\frac{y^{\alpha_2}\, z^{\beta_1}}{x^{\gamma_1-1}}\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_m}{m!}\frac{(\alpha_2)_{n+p}}{n!}\frac{(\beta_2)_n\,(\beta_1)_{m+p}}{p!\,(\gamma_1)_{m+n+p}}\left(\frac{z}{x}\right)^m\left(\frac{y}{x}\right)^n\left(\frac{yz}{x}\right)^p$$

Hence

(2·2) $$\left(1+\frac{\phi}{\Omega}\right)^{-\alpha_1}\left(1+\frac{\omega}{\Omega}\right)^{-\beta_2}\left\{\frac{y^{\alpha_2}\, z^{\beta_1}}{x^{\gamma_1-1}}\, {}_2F_1\left(\alpha_2, \beta_1 ; \gamma_1 ; \frac{yz}{x}\right)\right\}$$

$$=\frac{y^{\alpha_2}\, z^{\beta_1}}{x^{\gamma_1-1}}F_T\left(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_1, \gamma_1 ; \frac{z}{x}, \frac{y}{x}, \frac{yz}{x}\right),$$

if $\left|\frac{z}{x}\right| < r, \left|\frac{y}{x}\right| < s, \left|\frac{yz}{x}\right| < t < 1$, then $r+s = rs+t$.

Now operating on $\left\{\frac{x^{\alpha_1}\ z^{\beta_3}}{y^{\beta_2-1}}\left(1-\frac{x}{y}\right)^{-\alpha_1}\right\}$ by the operator $e^{\frac{-\phi\Omega}{\omega}}\left(1+\frac{\Omega}{\omega}\right)^{-\beta_1}$

we get

$$(2\cdot3)\qquad e^{\frac{-\phi\Omega}{\omega}}\left(1+\frac{\Omega}{\omega}\right)^{-\beta_1}\left\{\frac{x^{\alpha_1}\ z^{\beta_3}}{y^{\beta_2-1}}\left(1-\frac{x}{y}\right)^{-\alpha_1}\right\} = \frac{x^{\alpha_1} z^{\beta_3}}{y^{\beta_2-1}}$$

$$F_D\left(\alpha_1, \beta_1, \beta_2, \beta_3\,;\, \beta_2\,;\, \frac{x}{y}, \frac{x}{y}, \frac{xz}{y}\right) \left|\frac{x}{y}\right| < 1, \left|\frac{xz}{y}\right| < 1.$$

We also obtain the following results :

$$(2\cdot4)\qquad {}_0F_1(-\,;\,\gamma_3\,;\,\Omega\phi)\ {}_1F_1(\beta_1\,;\,\gamma_1\,;\,\Omega)\{x^{\alpha_1} z^{\gamma_2}(1-xz)^{-\alpha_1}\}$$

$$= x^{\alpha_1} z^{\gamma_2} F\ (\alpha_1, \alpha_1, \alpha_1, \beta_1, \gamma_2, \gamma_2\,;\, \gamma_1, \gamma_2, \gamma_3\,;\, x, xz, xz),$$

if $\quad |x| < r,\ |xz| < s$, then $r+4s = 1$.

$$(2\cdot5)\qquad {}_0F_1(-\,;\,\gamma_1\,;\,\Omega\phi)\left(1+\frac{\Omega}{\omega}\right)^{-\beta_2}\left\{\frac{x^{\alpha_1}\ z^{\beta_1}}{y^{\gamma_2-1}}\ {}_2F_1\left(\alpha_1, \beta_1\,;\,\gamma_2\,;\,\frac{xz}{y}\right)\right\}$$

$$= \frac{x^{\alpha_1}\ z^{\beta_1}}{y^{\gamma_2-1}} F_F\left(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1\,;\,\gamma_1, \gamma_2, \gamma_2\,;\, xz, \frac{x}{y}, \frac{xz}{y}\right),$$

if $\quad |xy| < r, \left|\frac{x}{y}\right| < s, \left|\frac{xz}{y}\right| < t$ then $rs = (1-s)(s-t)$.

$$(2\cdot6)\qquad {}_0F_1(-\,;\,\gamma_3\,;\,\Omega\phi)\ {}_1F_1(\alpha_1\,;\,\gamma_1\,;\,\Omega)\{x^{\beta_1} z^{\gamma_2}(1-z)^{-\beta_2}\}$$

$$= x^{\beta_1} z^{\gamma_2} F_K\ (\alpha_1, \gamma_2, \gamma_2, \beta_1, \beta_2, \beta_1\,;\,\gamma_1, \gamma_2, \gamma_3\,;\, x, z, xz),$$

if $\quad |x| < r,\ |z| < s,\ |xz| < t$ then $(1-r)(1-s) = t$.

$$(2\cdot7)\qquad {}_0F_1(-\,;\,\gamma_1\,;\,\Omega\phi)\ {}_2F_0\left(-\beta_2, \alpha_2\,;\,-\,;\,-\frac{1}{\omega}\right)\left\{\frac{x^{\alpha_1} z^{\beta_1}}{y^{\gamma_2}-1}\ {}_2F_1(\alpha_1, \beta_1\,;\,\gamma_2\,;\,xz)\right\}$$

$$= \frac{x^{\alpha_1} z^{\beta_1}}{y^{\gamma_2}-1}\left(F_R\left(\alpha_1, \alpha_2, \alpha_1, \beta_1, -\beta_2, \beta_1\,;\,\gamma_1, \gamma_2, \gamma_2\,;\, xz, \frac{1}{y}, \frac{xz}{y}\right),\right.$$

where β_2 is *an* $+$ *ve* integer, and if

$$|xz| < r, \left|\frac{1}{y}\right| < s, \left|\frac{xz}{y}\right| < t \text{ then } s(1\sqrt{r})^2 + t(1-s) = 0$$

$$(2\cdot8)\qquad {}_2F_0\left(\alpha_1, -\beta_1\,;\,-\,;\,-\frac{1}{\Omega}\right)\left(1+\frac{\omega}{\Omega}\right)^{-\beta_2}\left\{\frac{y^{\alpha_2}}{x^{\gamma_1-1}}\ {}_2F_1\left(\alpha_2, \beta_3\,;\,\gamma_1\,;\,\frac{y}{x}\right)\right\}$$

$$= \frac{y^{\alpha_2}}{x^{\gamma_1 - 1}} F_8 \left(\alpha_1, \alpha_2, \alpha_2, -\beta_1, \beta_2, \beta_3 \,;\, \gamma_1, \gamma_1, \gamma_1 \,;\, \frac{1}{x}, \frac{y}{x}, \frac{y}{x} \right),$$

where β_1 is *an* $+$ *ve* integer, and if

$$\left|\frac{1}{x}\right| < r, \left|\frac{y}{x}\right| < s \text{ then } r+s = rs.$$

(2·9) $$e^{\frac{-\phi\,\Omega}{\omega}} \left(1+ \frac{\Omega}{\omega}\right)^{-\beta_2} \left\{ \frac{x^{\alpha_1}\, z^{\beta_3}}{y^{\gamma_2 - 1}} \, {}_2F_1(\beta_1, \alpha_1 \,;\, \gamma_1 \,;\, x) \right\}$$

$$= \frac{x^{\alpha_1}\, z^{\beta_3}}{y^{\gamma_2 - 1}} F_G \left(\alpha_1, \beta_1, \beta_2, \beta_3 \,;\, \gamma_1, \gamma_2 \,;\, x, \frac{x}{y}, \frac{xz}{y} \right),$$

if $\quad |x| < r, \left|\frac{x}{y}\right| < s$, and $\left|\frac{xz}{y}\right| < t$, then $r+s = 1$, and $r+t = 1$.

(2·10) $$\left(1+ \frac{\Omega}{\omega}\right)^{-\beta_2} e^{\frac{-\phi\,\Omega}{\omega}} \left\{ \frac{x^{\alpha_2}\, z^{\beta_1}}{y^{\gamma_2 - 1}} \, {}_2F_1(\beta_1, \alpha_1 \,;\, \gamma_1 \,;\, z) \right\},$$

$$= \frac{x^{\alpha_2}\, z^{\beta_1}}{y^{\gamma_2 - 1}} F_M \left(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1 \,;\, \gamma_1, \gamma_2, \gamma_2 \,;\, z, \frac{x}{y}, \frac{xz}{y} \right\}$$

if $\quad |z| < r, \left|\frac{x}{y}\right| < s, \left|\frac{xz}{y}\right| < t$, then $r+t = 1, s = 1$.

(2·11) $$\left(1+ \frac{\Omega}{\omega}\right)^{-\beta_2} \left(1+ \frac{\phi}{\omega}\right)^{-\alpha_3} \left\{ \frac{x^{\alpha_2}\, z^{\beta_1}}{y^{\gamma_2 - 1}} \, {}_2F_1(\beta_1, \alpha_1 \,;\, \gamma_1 \,;\, z) \right\}$$

$$= \frac{x^{\alpha_2}\, z^{\beta_1}}{y^{\gamma_2 - 1}} F_N \left(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1 \,;\, \gamma_1, \gamma_2, \gamma_2, \,;\, z, x/y, z/y \right),$$

if $\quad |z| < r, \left|\frac{x}{y}\right| < s, \left|\frac{z}{y}\right| < t$, then $s(1-r) + t(1-s) = 0$.

(2·12) $$e^{-\frac{\theta\omega}{\Omega}} \, {}_1F_1 \left[\beta_1 \,;\, \gamma_1 \,;\, \eta\phi\right] \left\{ \frac{y^{\alpha_2}\, z^{\alpha_1}\, \eta^{\beta_1}}{x^{\gamma_2 - 1}} \, {}_2F_1 \left(\alpha_1, \beta_2 \,;\, \gamma_2 \,;\, \frac{z}{x}\right) \right\}$$

$$= \frac{y^{\alpha_2}\, z^{\alpha_1}\, \eta^{\beta_1}}{x^{\gamma_2 - 1}} F_P \left(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2 \,;\, \gamma_1, \gamma_2, \gamma_2 \,;\, \eta z, \frac{y\eta}{x}, \frac{z}{x} \right),$$

if $\quad |z\eta| < r, \left|\frac{\eta y}{x}\right| < s, \left|\frac{z}{x}\right| < t$ then $(st - s - t)^2 = 4rst$.

(2·13) $${}_1F_1(\alpha_1 \,;\, \gamma_1 \,;\, \Omega)\, {}_1F_1(\alpha_2 \,;\, \gamma_2 \,;\, \omega) \left\{ x^{\beta_1}\, y^{\beta_2}\, {}_2F_1(\beta_3, \alpha_3 \,;\, \gamma_3 \,;\, z) \right\}$$

$$= x^{\beta_1} y^{\beta_2} \varphi_1 (\alpha_1, \alpha_2, \alpha_3 ; \beta_1, \beta_2, \beta_3 ; \gamma_1, \gamma_2, \gamma_3 ; x, y, z) \; |x| < 1, |y| < 1, |z| < 1,$$

(2·14) $$\left(1+\frac{\Omega}{\phi}\right)^{-\alpha_1} \left(1+\frac{\omega}{\phi}\right)^{-\beta_2} \left\{ \frac{x^{\beta_1} y^{\alpha_2}}{z^{\gamma_1 - 1}} {}_2F_1\left(\alpha_3, \beta_3 ; \gamma_1 ; \frac{1}{z}\right)\right\}$$

$$= \frac{x^{\beta_1} y^{\alpha_2}}{z^{\gamma_1-1}} \varphi_2\left(\alpha_1, \alpha_2, \alpha_3 ; \beta_1, \beta_2, \beta_3 ; \gamma_1 : \frac{x}{z}, \frac{y}{z}, \frac{1}{z}\right) \left|\frac{x}{z}\right| < 1, \left|\frac{y}{z}\right| < 1, |z| > 1,$$

(2·15) $${}_1F_1(\alpha_1; \gamma_1; \Omega) \left(1+\frac{\omega}{\phi}\right)^{-\beta_2} \left\{ \frac{x^{\beta_1} y^{\alpha_2}}{z^{\gamma_2-1}} {}_2F_1\left(\alpha_3, \beta_3 ; \gamma_2 ; \frac{1}{z}\right)\right\}$$

$$= \frac{x^{\beta_1} y^{\alpha_2}}{z^{\gamma_2-1}} \varphi_3 \left(\alpha_1, \alpha_2, \alpha_3 ; \beta_1, \beta_2, \beta_3 ; \gamma_1, \gamma_2 ; x, \frac{y}{z}, \frac{1}{z}\right) |x| < 1, \left|\frac{y}{z}\right| < 1, |z| > 1.$$

(2·16) $$\left(1+\frac{\Omega}{\phi}\right)^{-\alpha_1} e^{\frac{-\omega\Omega}{\phi}} \left\{ \frac{x^{\beta_1} y^{\alpha_2}}{z^{\gamma_1-1}} {}_2F_1\left(\alpha_3, \beta_1 ; \gamma_1 ; \frac{x}{z}\right)\right\}$$

$$= \frac{x^{\beta_1} y^{\alpha_2}}{z^{\gamma_1-1}} \varphi_4 \left(\alpha_1, \alpha_2, \alpha_3 ; \beta_1, ; \gamma_1 ; \frac{x}{z}, \frac{xy}{z}, \frac{x}{z}\right) \left|\frac{x}{z}\right| < 1, \left|\frac{xy}{z}\right| < 1.$$

(2·17) $$e^{\frac{-\omega\Omega}{\phi}} {}_1F_1(\alpha_1 ; \gamma_1 ; \Omega) \left\{ \frac{x^{\beta_1} y^{\alpha_2}}{z^{\gamma_2-1}} {}_2F_1\left(\alpha_3, \beta_1 ; \gamma_2 ; \frac{x}{z}\right)\right.$$

$$= \frac{x^{\beta_1} y^{\alpha_2}}{z^{\gamma_2-1}} \varphi_5 \left(\alpha_1, \alpha_2, \alpha_3 ; \beta_1 ; \gamma_1, \gamma_2 ; x, \frac{xy}{z}, \frac{x}{z}\right) |x| + \left|\frac{xy}{z}\right| < 1, |y| > 1.$$

(2·18) $${}_1F_1(\alpha_1 : \gamma_1 ; \Omega)\, {}_1F_1(\alpha_2 ; \gamma_2 ; \omega) \{ x^{\beta_1} y^{\beta_2} {}_2F_1(\beta_2, \alpha_3 ; \gamma_3 ; y)\}$$

$$= x^{\beta_1} y^{\beta_2} \varphi_6(\alpha_1, \alpha_2, \alpha_3 ; \beta_1, \beta_2 ; \gamma_1, \gamma_2, \gamma_3 ; x, y, y) \quad |x| < 1, |y| < \tfrac{1}{2}.$$

(2·19) $$e^{\frac{-\omega}{\phi}} {}_2F_1(\beta_1, \alpha_1 ; \gamma_1 ; \Omega) \left\{ \frac{x^{\beta_2} y^{\alpha_2}}{z^{\gamma_2-1}} {}_2F_1\left(\alpha_3, \beta_2 ; \gamma_2 ; \frac{x}{z}\right)\right\}$$

$$= \frac{x^{\beta_2} y^{\alpha_2}}{z^{\gamma_2-1}} \varphi_7\left(\alpha_1, \alpha_2, \alpha_3 ; \beta_1, \beta_2 ; \gamma_1, \gamma_2 ; x, \frac{y}{z}, \frac{x}{z}\right) |x| < 1, \left|\frac{y}{z}\right| < 1, \left|\frac{x}{z}\right| < 1.$$

(2·20) $$e^{\frac{-\omega\Omega}{\phi}} {}_2F_1(\alpha_1, \beta_1 ; \gamma_1 ; \xi) \left\{ \frac{x^{\beta_2} y^{\alpha_2}}{z^{\gamma_2-1}} {}_2F_1\left(\alpha_2, \beta_2 ; \gamma_2 ; \frac{xy}{z}\right)\right\}$$

$$= \frac{x^{\beta_2} y^{\alpha_2}}{z^{\gamma_2-1}} \varphi_8\left(\alpha_1, \alpha_2 ; \beta_1, \beta_2 ; \gamma_1, \gamma_2 ; \xi, \frac{xy}{z}, \frac{xy}{z}\right), |\xi| < 1, \left|\frac{xy}{z}\right| < \tfrac{1}{2}.$$

$$(2\cdot21)\qquad e^{\frac{-2\omega\Omega}{\phi}}\left\{\frac{x^{\beta_1}\,y^{\alpha_1}}{z^{\gamma_1-1}}\;{}_2F_1\left(\alpha_1,\beta_1;\gamma_1;\frac{xy}{z}\right)\right\}=\frac{x^{\beta_1}\,y^{\alpha_1}}{z^{\gamma_1-1}}\;\phi_2\left(\alpha_1;\beta_1;\gamma_1;\frac{xy}{z},\frac{xy}{z},\frac{xy}{z}\right)$$

where $\left|\frac{xy}{z}\right|<1.$

$$(2\cdot22)\qquad {}_0F_1(-;\gamma_3;\omega\phi)\;{}_1F_1(\alpha_1;\gamma_1;y\Omega)\left\{x^{\beta}\,y^{\alpha}\,z^{\gamma_2}\,(1-xz)^{\beta}\right\}$$

$$=x^{\beta}\,y^{\alpha}\,z^{\gamma_2}\,H_B\,(\alpha,\beta,\gamma_2;\gamma_1,\gamma_2,\gamma_3;xy,xz,yz)$$

if $\quad |xy|<r,\ |xz|<s,\ |yz|<t$, then $r+s+t+2\sqrt{rst}=1.$

$$(2\cdot23)\qquad e^{-\frac{\theta\omega}{\Omega}}\left(1+\frac{\eta\phi}{\Omega}\right)^{-\beta}\left\{\frac{y^{\beta'}\,z^{\alpha}}{x^{\gamma_1-1}}\;\eta^{\beta}\;{}_2F_1\left(\beta',\alpha;\gamma_1;\frac{yz}{x}\right)\right\}$$

$$=\frac{\eta^{\beta}\,y^{\beta'}}{x^{\gamma_1-1}}\,z^{\alpha}\,H_C\left(\alpha,\beta,\beta';\gamma_1;\frac{z\eta}{x},\frac{y\eta}{x},\frac{yz}{x}\right),$$

and

$$(2\cdot24)\qquad e^{\frac{-\theta\omega}{\Omega}}\;{}_1F_1(\beta;\gamma;\eta\phi)\left\{\frac{y^{\beta'}\,z^{\alpha}}{x^{\gamma'-1}}\;\eta^{\beta}\;{}_2F_1\left(\beta',\alpha;\gamma';\frac{yz}{x}\right)\right\}$$

$$=\frac{y^{\beta'}\,\eta^{\beta}\,z^{\alpha}}{x^{\gamma'-1}}\;H_A\left(\alpha,\beta,\beta';\gamma,\gamma';\eta z,\frac{y\eta}{x},\frac{yz}{x}\right)$$

if $\quad |\eta z|<r,\ \left|\frac{y\eta}{x}\right|<s,\ \left|\frac{yz}{x}\right|<t$ then $r+s+t=1+st,$

where the operators in (2·23) and (2·24) are not commutative.

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Reactions between Chromium and Iron salts and Alkali

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Abstract

The stoichiometry of these reactions in very dilute solutions has been studied. The results show that when the alkali is added to dilute solutions of $Cr(NO_3)_3$, $CrCl_3$ or $FeCl_3$ (0·05N or less), the anions are replaced progressively by hydroxyl. For example, one gets $Cr(OH)Cl_2$, $Cr(OH)_2Cl$ and $Cr(OH)_3$. When these salts are added to excess dilute NaOH, one of course gets $Cr(OH)_3$ or $Fe(OH)_3$. Some deviation from exact stoichiometry is caused, perhaps by peptization of the chromium hydroxide by excess salt. The excess NaOH in these mixtures can be titrated exactly with dilute HCl. In the case of chromium the amount of the precipitate should be kept small, otherwise errors amounting to 3–5% are caused perhaps due to formation of $Cr(OH)Cl_2$.

Introduction

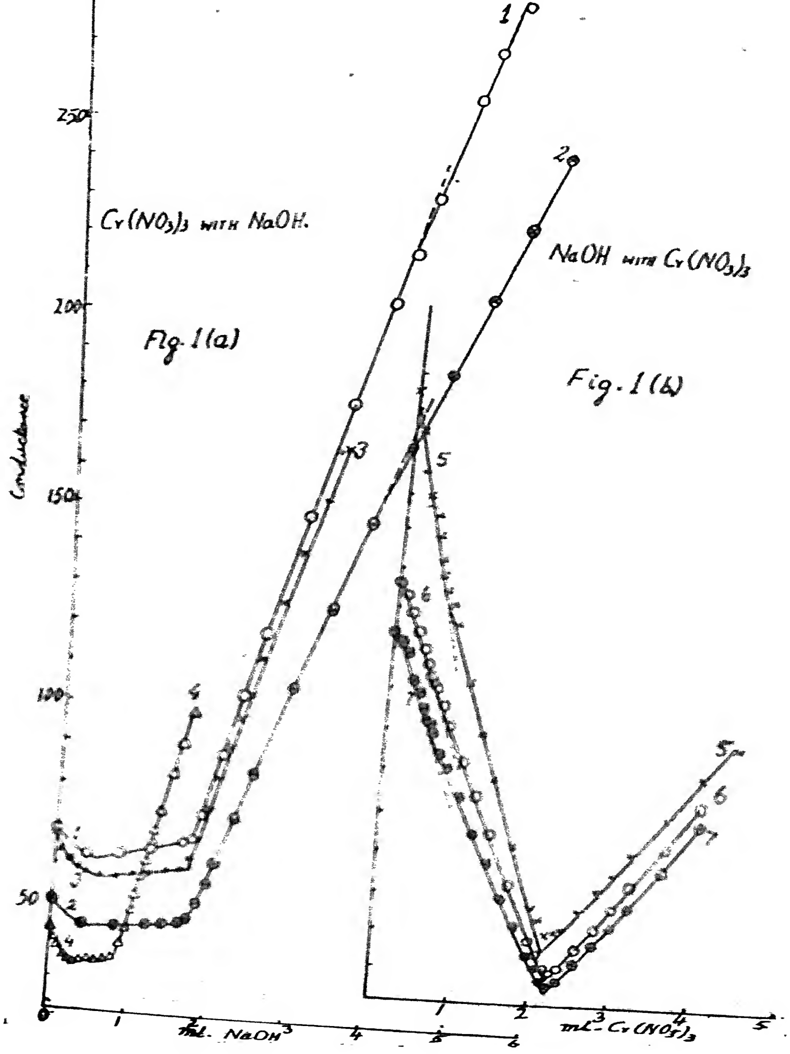
We have studied the reactions between zinc and aluminium salts and alkali in earlier papers[1]. We found that exact stoichiometry in some cases was disturbed due to adsorption and formation of basic salts. Tiwari and Ghosh[5] have indicated the possibility of formation of $Cr(OH)_2Cl$ and $Cr(OH)Cl_2$ in mixtures of $CrCl_3$ and NaOH studied conductometrically. We have indicated the formation of similar compounds in our study on hydrolysis of ferric chloride[2]. Much of the earlier work on mixtures of iron and chromium salts and alkali is concerned with the formation of colloidal hydroxides and the nature of particles *e.g.* those of Konche[7], Raikov[8], Gustavson[9] Daruwalla and Nabar[10], Lodha,[11] Tiwari and Ghosh[6] and Ermolenko[12].

Our previous experience as well as some references in the literature show that ferric hydroxide may be peptized by ferric and chromic salts and with acids including acetic.[3] When alkali is in the titration cell and these reagents are in the burette,, peptization will raise the right hand portion of the titration graph with the possibility of depressing the apparent salt requirement for the alkali.

Experimental

Stock solutions of analytical grades of $Cr(NO_3)_3$, $CrCl_3$, $FeCl_3$, NaOH and HCl were prepared and carefully standardized by gravimetric and volumetric methods. Each of the salts was checked for excess acid and found satisfactory. Freshly prepared conductivity water was used throughout this work, The glass-ware was mostly pyrex or well weathered ordinary glass. Special care was taken to maintain the temperature of the thermostat constant to within $+-0{\cdot}05°$ C. Other details will be found in our earlier works (op. cit.).

The titrations were carried out in both directions at several dilutions of the reagents. The results are summarised in Tables I, II and III. The titration graphs giving more details are shown in Figs. 1 to 4. The data have been corrected for progressive dilution.



Cr(NO₃)₃ with NaOH.
NaOH with Cr(NO₃)₃
Fig. 1(a)
Fig. 1(b)
Conductance
ml. NaOH
ml. Cr(NO₃)₃

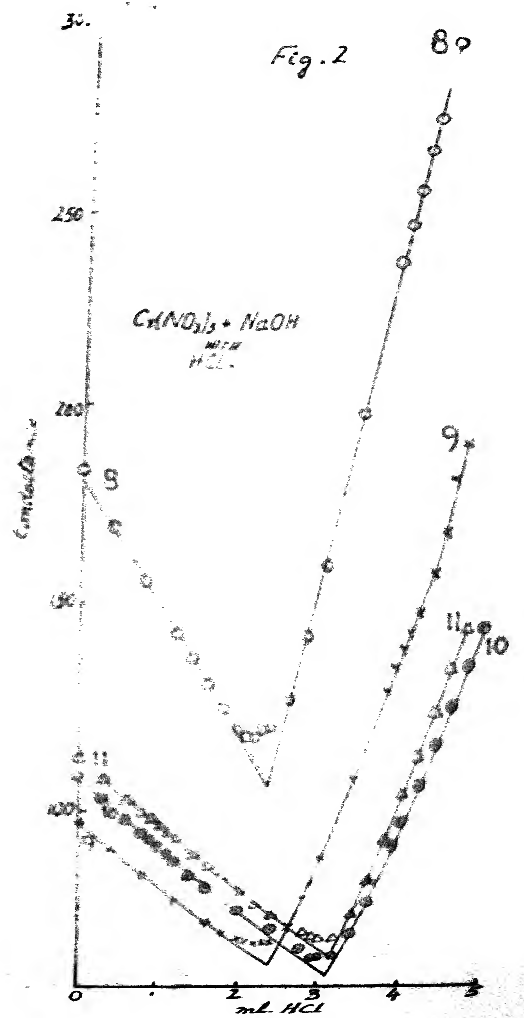


Fig. 2
Cr(NO₃)₃ + NaOH with HCl.
Conductance
ml. HCl

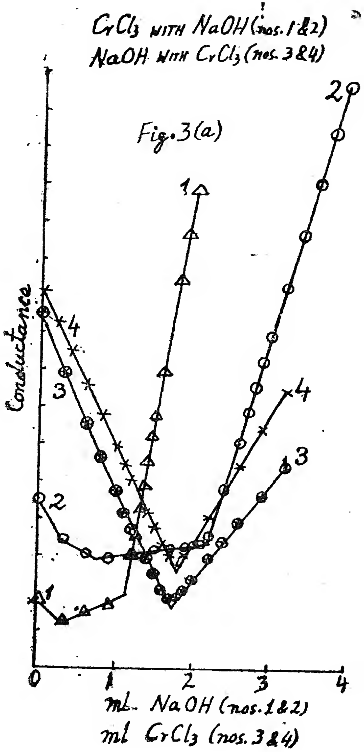


Fig. 3(a)

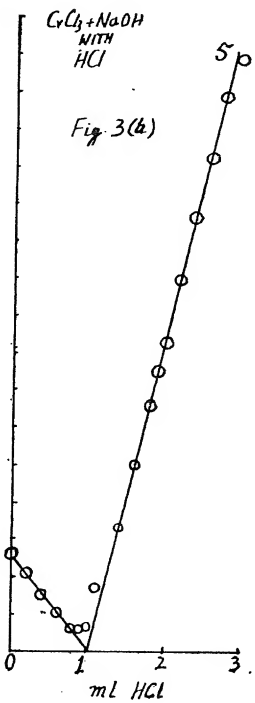


Fig. 3(b)

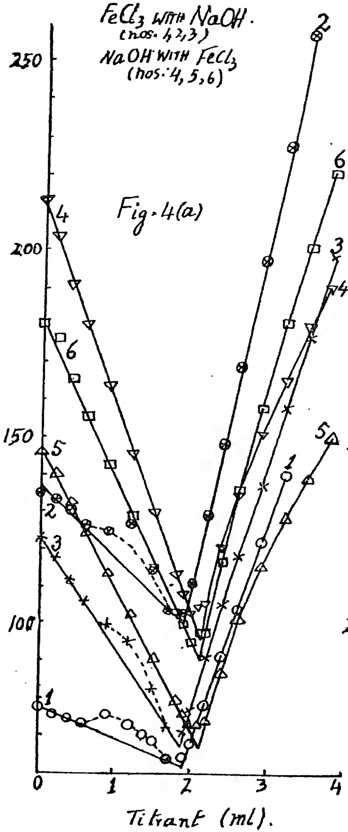


Fig. 4(a)

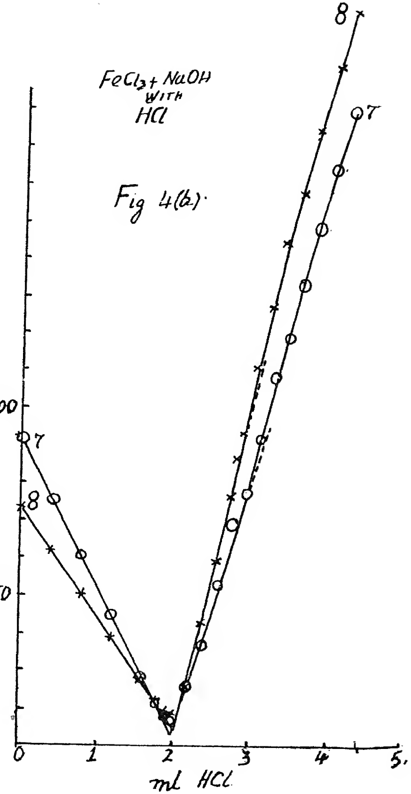


Fig 4(b)

TABLE I

Reactions of $Cr(NO_3)_3$

	Cell	Burette	Infl. found		Remarks	
1.	0·4586N $Cr(NO_3)_3$ 4 ml + water 36 ml	1·078N NaOH	0·60 ml 1·70 ml	20°C.	Req. for $Cr(OH)(NO_3)_2$ Req. for $Cr(OH)_3$	0·57 ml 1·70 ml
2.	do + water 76 ml	do	0·60 ml 1·70 ml	21°C.	Req. for $Cr(OH)(NO_3)_2$ Req. for $Cr(OH)_3$	0·57 ml 1·70 ml
3.	do + water 156 ml	do	0·60 ml 1·675 ml	20°C	do	
4.	do 2 ml + water 158 ml	do	0·30 ml 0·85 ml	20°C	Req. for $Cr(OH)(NO_3)_2$ Req. for $Cr(OH)_3$	0·284 ml 0·851 ml
5.	1·078N NaOH 1 ml + water 19 ml	0·4586N $Cr(NO_3)_3$	0·60 ml 2·20 ml	20°C	Req for $Cr(OH)_3$ Req. for $Cr(OH)(NO_3)_2$	0·57 ml 2·35 ml
6.	do 1 ml + water 79 ml	do	0·60 ml 2·20 ml	20°C	Req. for $Cr(OH)_3$ Req for $Cr(OH)(NO_3)_2$	0·57 ml 2·35 ml
7.	do 1 ml + water 159 ml	do	0·55 ml 2·25 ml	20°C	do	
8.	0·4586N $Cr(NO_3)_3$ 4 ml + 1·078N NaOH 4 ml + water 72 ml	1·077N HCl	2·40 ml	25°C	Req. for $Cr(OH)_3$	2·303 ml
9.	do + water 152 ml	do	2·40 ml	25°C	do	
10.	0·4586N $Cr(NO_3)_3$ 2 ml + 1·078N NaOH 4 ml + water 74 ml	do	3·125 ml	25°C	Req. for $Cr(OH)_3$	3·150 ml
11.	do + water 154 ml	do	3·150 ml	25°C	Req. for $Cr(OH)_3$	3·150 ml

TABLE II

Reactions of $CrCl_3$

	Cell	Burette	Infl. found		Remarks	
1.	0·590N $CrCl_3$ 2 ml + w. 18 ml	1·070N NaOH	0·33 ml 1·10 ml	34°C	Req for $Cr(OH)Cl_2$ Req. for $Cr(OH)_3$	0·34 ml 1·10 ml
2.	0·590N $CrCl_3$ 2 ml + w. 78 ml	0·535 NaOH	0·73 ml 2·20 ml	35°C	Req. for $Cr(OH)Cl_2$ Req. for $Cr(OH)_3$	0·73 ml 2·20 ml
3.	0·535N NaOH 2 ml + w. 18 ml	0·590N $CrCl_3$	1·15 ml 1·75 ml	35°C	Req. for $NaCrO_2$ Req. for $Cr(OH)_3$	1·17 ml 1·81 ml
4.	0·535N NaOH 2 ml + w. 78 ml	do	1·80 ml	35°C	Req. for $Cr(OH)_3$ $NaCrO_2$ not observed due to high dilution.	1·81 ml
5.	0·590N $CrCl_3$ 2 ml + ·535N NaOH 4 ml + w. 34 ml	1·077N HCl	1·02 ml	35°C	Req. 1·256, 0·95, 0·891 and 0·526 ml for $Cr(OH)_2Cl$, $Cr(OH)Cl_2$, $Cr(OH)_3$ and $NaCrO_2$ resp.	

TABLE III

Reactions of Ferric chloride

	Cell	Burette	Infl. found		Remarks	
1.	1·021N $FeCl_3$ 2 ml + w. 38 ml	1·07N NaOH	0·60 ml 1·90 ml	30°C	Req. for $Fe(OH)Cl_2$ Req. for $Fe(OH)_3$	0·63 ml 1·91 ml
2.	do + w. 78 ml	do	do	do	do	
3.	do + w. 158 ml	do	do	do	do	
4.	1·07N NaOH 2 ml + w. 38 ml	1·021N $FeCl_3$	2·125 ml	do	Req. for $Fe(OH)_3$	2·10 ml
5.	1·07N NaOH 2 ml + w. 78 ml	do	do	do	do	
6.	do + w. 158 ml	do	do	do	do	
7.	1·021N $FeCl_3$ 1 ml + 1·07N NaOH 3 ml + w. 16 ml	1·077N HCl	2·00 ml 3·00 ml	do do	Req. for $Fe(OH)_3$ Req. for all NaOH	2·025 ml 2·98 ml
8.	do + do + w. 100 ml	do	do	do	do	

Discussion

When NaOH is added to solutions of $Cr(NO_3)_3$, $CrCl_3$ or $FeCl_3$ there is a fall in conductivity. This fall is not as strong as it should be if hydrogen ions coming from the acid liberated by hydrolysis were being removed. As the initial dilution of the salt is increased the initial slope becomes steeper because the fraction of salt removed by precipitation by the same amount of alkali delivered increases, Figs. 1*a*, 3*a* and 4*a*.

The initial fall is followed by a break at about onethird of the alkali required for the normal hydroxide, indicating the formation of $Cr(OH)\ (NO_3)_2$, $Cr(OH)Cl_2$ and $Fe(OH)Cl_2$. The conductivity then increases slightly in case of chromium salts (and ferric chloride only when the initial concentration is not too low, Curve 1, Fig. 4*a*). With dilute ferric chloride the break is followed by a loop with progressive fall in conductivity. In all cases this portion of the graph ends with a break at one equivalent alkali added for an equivalent of salt, indicating the formation of normal hydroxides, $Cr(OH)_3$ or $Fe(OH)_3$. The stoichiometry is exact. The conductivity then increases rapidly due to hydroxyl ions. We presume that the middle portion of these graphs is due to progressive formation of $Cr(OH)_2\ NO_3$, $Cr(OH)_2\ Cl$ and $Fe(OH)_2\ Cl$. One should expect that these particles should be less conducting than the corresponding monohydroxy compounds unless the latter are more hydrated. It is significant that the increase in conductivity is larger when $CrCl_3$ is taken in place of $Cr(NO_3)_3$. Nitrates as a rule show reluctance towards hydration.

The quantitative results stated above regarding formation of particles like $Cr(OH)Cl_2$ and $Cr(OH_2)\ Cl$ receive further confirmation from the visual observation that in the direct titration of chromium salts the precipitate did not appear till one equivalent of alkali had been added. In the case of ferric chloride the dark brown precipitate appeared with the first drop of alkali. It seems that these basic salts of chromium are soluble but those of iron are not.

In the reverse titrations (alkali in the cell) the only sharp break corresponds to formation of the normal hydroxides. After this break the increase in conductivity is so large that the possibility of detecting the stages like $Cr(OH) Cl_2$ becomes remote. In these titrations we note that the stoichiometry of the formation of $Fe(OH)_3$ is exact but that corresponding to $Cr(OH)_3$ is not. The chromium requirement is somewhat less than the calculated value, Tables I and II. One may think that a small quantity of alkali is adsorbed by the freshly precipitated $Cr(OH)_3$, so that the right hand portions of the titration graphs (Figs. 1*b*, 3*b*) are raised, depressing the apparent chromium requirement. The peptization is unlikely to be preceded by the formation of $Cr(OH)_2Cl$, since that would raise the chromium requirement a little.

When mixtures of these salts and alkali are back titrated with HCl, the result is again very satisfactory with ferric chloride, and the $Fe(OH)_3$ stage is accurately determined (Table III, Nos. 7 & 8), but with chromium 3-4% more acid is consumed than required for the $Cr(OH)_3$ stage. We presume that a little $Cr(OH)_2$ or $Cr(OH)_2NO_3$ is formed. Free HCl or HNO_3 and not the corresponding salts (NaCl or $NaNO_3$) are necessary for the formation of these compounds.

The $CrCl_3$-NaOH and $Cr(NO_3)_3$-NaOH mixtures give an initial break when titrated with HCl. It corresponds nearly to consumption of alkali one-third over and above that required for the normal hydroxide. This signifies the formation of compounds like $NaCr(OH)_4$ *i.e.* $NaCrO_2$. This break is not observed with ferric chloride. We have reported the formation of $NaAlO_2$ in a previous paper[4]. One might expect to find such breaks in the direct titration of chromium salts with alkali. Actually, the conductivity of the mixture at the stage where these breaks should be expected in these direct titration is already so high that they would be hard to locate.

Acknowledgements

The authors are grateful to Dr. H. G. Mishra for discussions and to Patna University for a research scholarship to one of us (B. R. G.).

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The thermal and photo chemical oxidation of alcohols by Potassium Dichromate
Part III The photo chemical oxidation of ethylene glycol

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[Received on 13th June, 1967]

Abstract

The photo chemical oxidation of ethylene glycol by potassium dichromate has been studied at two pH values. The initial velocity varies linearly with the alcohol and acid chromate ion concentration and also with the intensity of light. The second order constants show very little variation with acid concentration in this range. The influence of cerous ion on the oxidation has also been examined and implicates tetravalent chromium species in the rate determining step.

In part I[1] and II[2] of this series, the thermal oxidations of some alcohols and the photo chemical oxidation of ethyl alcohol respectively were reported. This paper presents the extension of the photo chemical studies to ethylene glycol.

The experimental techniques have been described elsewhere.[1,2] The studies were made at pH 4·0 and 4·76 using sodium acetate–acetic acid buffers. Temperature was kept constant at $35 \pm 0{\cdot}1°C$ and an ionic strength of 0·11 was also maintained.

Results

The initial velocity expressed as the rate of disappearance of Cr (VI) was found to increase linearly with the concentration of ethylene glycol and the acid

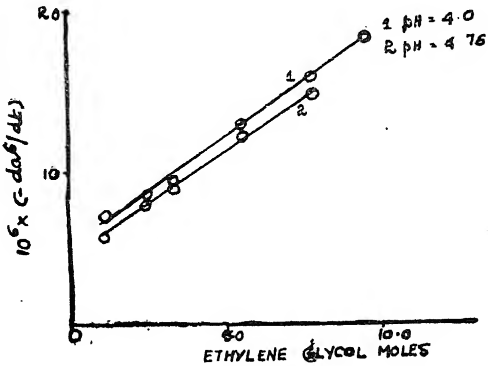

Fig. 1.

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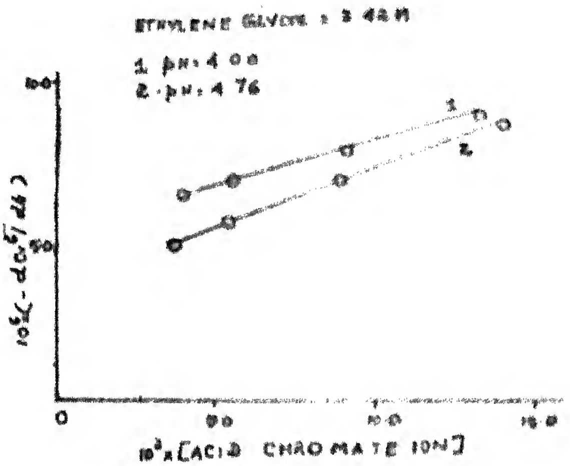
ETHYLENE GLYCOL = 3.48 M
1. pH: 4.00
2. pH: 4.76
10⁸x[ACID CHROMATE ION]

Fig. 2.

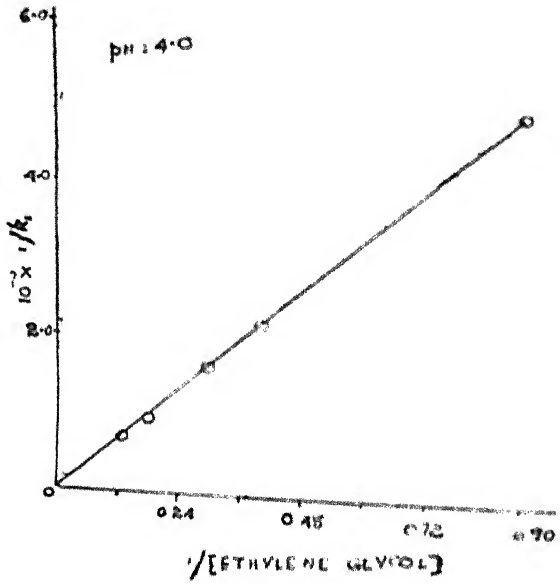
pH : 4.0
1/[ETHYLENE GLYCOL]

Fig. 3.

Chromate ion (Figs. 1 and 2). The calculation of the acid chromate ion concentration at various concentrations of hexavalent chromium has been described elsewhere.[2] As in the case of ethyl alcohol, at constant acid concentration the rate data were found to fit into the expression.

$$Z + \ln Z = -k_1 t + C \qquad (1)$$
$$\text{where} \quad Z = (1 + 8\,K_4\,Cr^{6+})^{\frac{1}{2}} - 1$$

The values of the pseudo first order rate constant, k_1, were obtained graphically from plots of $Z + \ln Z$ versus time. When the reciprocals of these values of k_1 were plotted against the reciprocal of the respective concentrations of ethylene glycol a linear dependence was observed (Fig. 3). This is reminiscent of the behaviour of ethyl alcohol. However, there was no intercept. The value of the second order rate constant, k_2, given by the reciprocal of the slope of such plots were found to be $1{\cdot}90 \times 10^{-4}$ and $1{\cdot}75 \times 10^{-4}$ l.mole^{-1} min^{-1} at pH 4·0 and 4·76 respectively.

The initial velocity was also found to vary linearly with the intensity of incident light. The data are given in Table 1.

TABLE 1

Variation of velocity with light intensity

$10^5 \times$ Intensity	$10^6 \times$ Initial velocity	Quantum Yield
2·19	16·32	0·74
1·83	13·98	0·76
1·26	9·48	0·75

Ethylene glycol = 5·70M ; Cr^{6+} = 0·0398 M ; pH = 4·0
Units : Intensity = Einsteins/litre/min ; Velocity = Moles/litre/min
Quantum Yield = Moles/Einstein

The influence of cerous ion on the photo chemical oxidation was also examined. The cerous ion was found to retard the rate of oxidation at much lower concentrations than that required for the corresponding dark reaction. The ratio of initial velocities in presence and absence of cerous ion decreased in an asymptotic manner with increase in cerous ion concentration. The lowest value observed was 0·44 and further increase of cerous ion concentration did not result in any significant increase in retardation of the rate. In other words the velocity is slightly more than halved in presence of optimum amounts of cerous ion. The relevant data can be found in Table 2.

TABLE 2

Influence of Cerous ion on the rate of oxidation

$10^4 \times$ Cerous ion Concn. Moles	V/V_0
Nil	1·00
1·00	0·70
1·50	0·60
2·25	0·54
3·00	0·49
3·50	0·47
4·75	0·44

Ethylene glycol = 5·70 M ; Cr^{6+} = 0·0398 M ; pH = 4·0
Intensity = $2{\cdot}19 \times 10^{-5}$ Einsteins/litre/min

Discussion

The linear dependence of the initial velocity on the acid chromate ion and the validity of the expression (1) both serve to underline the specific dependence of the reaction on the acid chromate ion. This is a characterestic of the ester mechanism proposed by Westheimer and Novick[3] for the oxidation of isopropyl alcohol. The linear dependence of the velocity on the concentration of ethylene glycol as well as Klanning's demonstration[4] of the existence of a 1 : 1 complex between an alcohol and an acid chromate ion supports this conclusion. The 1 : 1 complex is likely to be an ester anion. That excitation of this particle initiates the photo chemical process is indicated by the very small variation of the second order rate constant, k_2, when going over from pH 4·0 to 4·76.

The observed linear dependence on light intensity precludes a chain mechanism and suggests that the reaction is a one quantum process. The retarding influence of cerous ion clearly implicates a tetravalent chromium species in the rate determining step. As the rate is nearly halved it would appear that one molecule of chromium (VI) disappears before the rate determining step and one after that. In all these features the photo chemical oxidation of ethylene glycol closely parallels that of ethyl alcohol reported earlier. An additional interesting feature is that as in the thermal oxidations only one hydroxyl group appears to get oxidized. As a plot of $1/k_1$ against 1/[ethylene glycol concentration] is linear over an eight fold variation, it appears that neither water nor ethylene glycol is a likely agent for the deactivation of the excited ester anions.

Acknowledgement

The author thanks Dr. K. K. Rohatgi, Department of Chemistry, Jadavpur University, for guidance and helpful discussions during the course of this work.

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Isolation and Investigations on the Alkaloid from the Root of *Delphinium denudatum* Wall.

By

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[Received on 25th September, 1967]

Abstract

A diterpenoid alkaloid, $C_{25}H_{39}NO_6$, has been isolated from the root of *Delphinium denudatum* Wall. The alkaloid has been found to contain two hydroxyl, one acetyl, two methoxyl and one $>N-CH_2CH_3$ groups. On the basis of some chemical, analytical and spectroscopic results and analogy with some known structures of other alkaloids isolated from the plants of the genus, Delphinium, a tentative structure has been assigned to it.

Delphinium denudatum Wall. (*Hindi* : Nirbisi, N. O. Ranunculaceae) is a small plant found in the Himalayas from Kashmir to Kumaon at a height of 8000–12000 ft. The root of this plant is bitter ; it cures 'Kapha', 'Vata', diseases of blood, snake-bite, headache and scorpion-sting[1].

A fairly large number of closely related monobasic alkaloids have been isolated from this genus. On account of their high toxicity these alkaloids have been studied intensively. But, inspite of their intrinsic interest and the fact that the raw material is often available in abundance, a satisfactory elucidation of the structures has not so far been possible due to the considerable practical difficulties attending the investigations. However, it has now been made clear that these alkaloids are derived from complex polyhydric amino alcohols, probably based on the formula, $C_{19}H_{28}NH$, and thus, they are related, formally at least, to diterpenes[2]. Recently, the application of some physico-organic techniques, such as X-ray crystallography and infrared and nuclear magnetic resonance spectroscopy, along with oxidation studies have enabled the chemists to assign tentative structures to many of these bases. But, inspite of good amount of investigations directed to the study of Delphinium alkaloids, not much head-way could be possible in regard to the establishment of the structures. The structural study of these still invites the attention of chemists. In view of the need and importance for such a study, the investigations on the chemical study of the root of *Delphinium denudatum* were undertaken and the results are reported in the present paper.

Procedure and Results

Isolation of the alkaloid : 4 Kg. of the dried and powdered root were extracted with petroleum ether (40–60°) in a Soxhlet. The extract was kept overnight when a white crystalline solid mass settled down at the bottom of the flask. The white mass was filtered out and washed several times with cold petroleum ether. It was recrystallised from chloroform–petroleum ether when fine star-shaped crystals (Yield 1·60 gm) melting at 157–158°C were obtained. This compound gave positive tests for alkaloids.

Found	Calculated for $C_{23}H_{39}O_6N$
C=66·73%, H=8·76%,	C=66·8%, H=8·68%
N=3·41% (Dumas, method)	N=3·11%
Mol. wt.=442·5 (Semi-micro Rast)	Mol. wt.=449

$[\alpha]^{25}_{D}=+26·5$ (in $CHCl_3$)

Thus, the molecular formula of the base appears to be $C_{23}H_{39}O_6N$. It yielded a picrate, m. p. 199–200°C, and a diacetate, $C_{29}H_{43}NO_8$, m. p. 131–132°C.

Nature of Nitrogen atom of the alkaloid : The nitrogen was not detected by Kjeldahl's method and it is therefore taken to be present in a ring system. Herzig-Meyer hydriodic acid treatment of the base gave ethyl iodide showing the presence of a $N-CH_2CH_3$ group. This was also indicated by a triplet absorption peak at 67 cps in the N. M. R. spectrum of the base.

Nature of the Oxygen atoms of the alkaloid : The Zeisel method of estimation of alkoxyl groups showed the presence of two methoxyl groups in the base. The N. M. R. spectrum of the base has two singlet absorption peaks at 198 and 201 cps characteristic of $-OCH_3$ groups. The absorption peaks at 229 and 185·5 cps are indicative of the presence, respectively, of $-CH-OCH_3$ and $\geqslant C-CH_2OCH_3$ type structures. The absorption peaks at 1460, 1355 and 1365 cm^{-1} in the infrared spectrum of the base further confirm the presence of methoxyl groups.

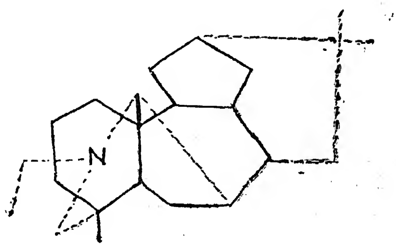
Wiesenberger's method[3] of estimation indicated the presence of one acetyl group in the base. This is supported by the appearance of absorption peaks at 124 and 291 cps in the N. M. R. spectrum, and at 1210 and 1180 cm^{-1} in the infrared spectrum of the base. A strong absorption peak at 1750 cm^{-1} in the infrared spectrum also appears. This can only be due to the carbonyl (C=O) group of the acetyl group and thus the presence of acetyl group receives further confirmation.

The base could neither be acetylated with acetic anhydride and pyridine nor with acetic anhydride and fused sodium acetate, thus indicating the absence of any primary alcoholic group. However, it gave a diacetate on acetylation with acetyl chloride which shows that two oxygen atoms are present as hydroxyl (–OH) groups. None of these –OH groups is phenolic as the base neither dissolves in sodium hydroxide nor does it give any coloration with ferric chloride. Hence, both –OH groups are alcoholic. The absorption peak at 1150 cm^{-1} in the I. R. spectrum suggests that at least one tertiary alcoholic ($\geqslant C-OH$) group is present in the molecule, whereas the peaks at 1275 and 1295 cm^{-1} indicate the presence of a secondary alcoholic ($>CH.OH$) group.

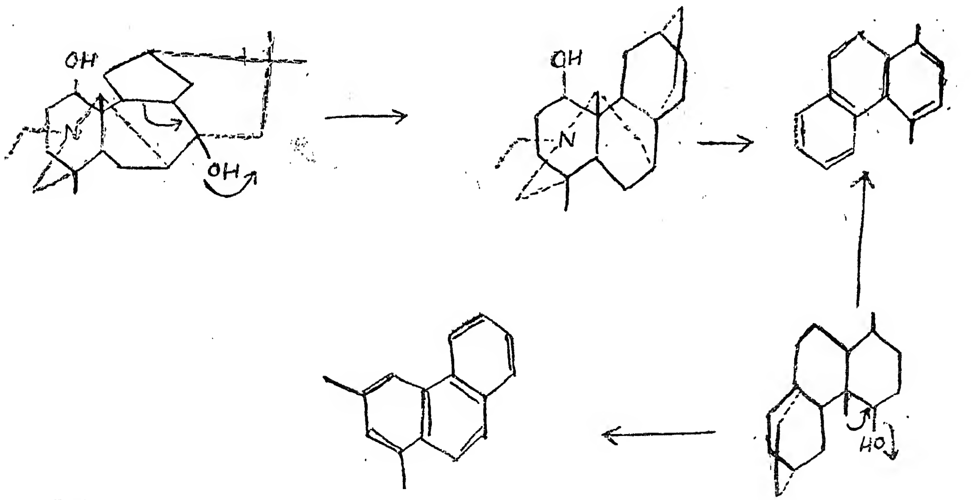
Thus, it is concluded that out of the six oxygen atoms two are present in the two methoxyl groups, two in the acetoxyl group and the remaining two as alcoholic groups, one secondary and the other tertiary.

Skeleton of the alkaloid : The absence of peak for unsaturation in the I. R. spectrum indicates that the base is saturated and therefore it should be hexacyclic. Knowing the nature and environment of hetero atoms and the functional groups, the formula of the alkaloid can be represented as $C_{19}H_{23}(OH)_2(\overset{O}{\overset{\|}{O C}}-CH_3)(OCH_3)_2 \cdot (NCH_2CH_3)$, which indicates the presence of a C(19) skeleton in the molecule.

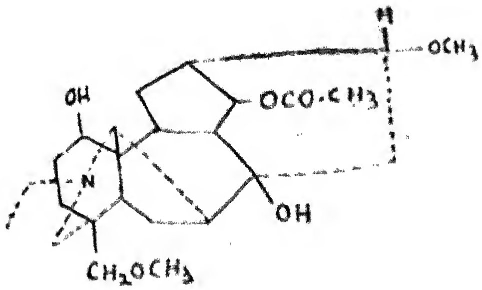
On dehydrogenation with selenium, 1 : 3 dimethyl phenanthrene was obtained. The presence of a C(19) skeleton and the formation of phenanthrene hydrocarbon on dehydrogenation suggests that the base, like other Delphinium alkaloid, has a diterpenoid frame-work. The high level of oxygen in the molecule further suggests its analogy with lycoctonine[4,5]. Therefore, the alkaloid seems to have a lycoctonine type skeleton given below :



The formation of 1 : 3 dimethyl phenanthrene on dehydrogenation can only be explained if the positions of the two -OH groups are assumed as shown below :



While this much of the study had been done, a paper by Pelletier and coworkers[6] appeared on the partial structure of 'condelphine'. The description given by them for their alkaloid was nearly the same as that of the alkaloid isolated by the authors. At our request, Dr. Pelletier kindly provided us with a sample of 'condelphine' isolated by him and it is now concluded by mixed melting point, thin layer chromatography and superimposable I. R. spectra that the alkaloid isolated by the authors is identical with condelphine. The probable structure which Dr. Pelletier and coworkers have suggested for condelphine is given below :



The N. M. R. and I. R. data and other studies made by the authors and discussed above also lend support to the same probable structure for the alkaloid. The degradative study of the alkaloid for confirming the structure is in progress.

Acknowledgement

Our thanks are due to Dr. S. N Khanna, N. C. L., Poona, for his valuable suggestions and for his help in the nuclear magnetic resonance spectroscopic studies.

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Chemical Examination of the Root of *Butea monosperma*

By

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[Received on 29th April, 1967]

Abstract

The root of *Butea monosperma* (*Hindi*: Dhak, Palas, N. O.—Leguminoseae) has been shown to contain glucose, glycine, a glycoside and an aromatic hydroxy compound.

Butea monosperma[1,2] is an Indian indigenous plant and occurs widely in the greater parts of India and Burma upto a height of 3000 feet. The root cures night blindness and other defects of sight and is useful in elephantiasis. It is also reported to make a woman sterile temporarily.

Experimental

About 10 kgs of defatted roots of *Butea monosperma* were exhaustively extracted with ethanol. The ethanolic extract was concentrated to half of its volume and kept overnight. A very small amount of orange yellow deposit settled at the bottom of the flask which was separated. This could not be studied because the amount obtained was very small and could not be purified.

The solvent from the ethanolic filtrate was removed by distilling it off and the concentrated filtrate was then transferred to a dish and kept on a water bath. A sticky mass was obtained which was then extrcted with warm water. The residue (A) and the aqueous extract (B) were studied separately.

To the aqueous extract (B) a saturated solution of lead acetate was added when a precipitate (lead-lake) appeared. The precipitate was filtered on a Buchner funnel and dried in an oven at 105–110°C.

Examination of the precipitate (Lead-lake): The deried and powdered lake was suspended in alcohol, dry hydrogen sulphide gas was passed into it and the precipitate filtered. Hydrogen sulphide from filtrate was removed by boiling. The filtrate was then concentrated on a water bath at 70–80°C, when a sticky residue was obtained. This residue gave positive tests for glycoside.

Study of the glycoside: The glycoside was hydrolysed by heating it with 50 ml of 2N–HCl over a steam bath for about 4 hours. The contents were diluted with distilled water when a precipitate was obtained. The precipitate (aglycone) was filtered and washed with distilled water.

The filtrate gave a positive test for reducing sugar, and on examination by paper and thin layer chromatography showed the presence of glucose.

The aglycone did not give the test for nitrogen, sulphur or phosphorus but showed the presence of phenolic and ketonic groups. It gave orange red colour with concentrated sulphuric acid, purple colour with aqueous NaOH, olive

brown colour with alcoholic $FeCl_3$ and no colour with Mg and HCl. Further study of the aglycone is in progress.

Examination of the filtrate

The filtrate from the lead-lake was examined for the presence of free amino acids and sugars. A qualitative study of amino acids and sugars was done by paper and thin layer chromatography

(*i*) *Examination of free sugars* : Two strips (12" × 2") of Whatman No. 1 paper were cut and one spot of the filtrate was placed on each strip. The chromatograms were developed by ascending technique in two different solvents.

(*a*) *n*–butanol : acetic acid : water (4 : 1 : 5) and

(*b*) *n*–butanol : ethanol : water : ammonia (45 : 5 : 49 : 1)

After development the chromatograms were sprayed with aniline hydrogen phthalate when a single spot was obtained. The RF value of sugar relating to the above spot was measured and compared with that given in the literature.[3,4]

Solvents

(*a*) *n*–Butanol	40%	(*b*) *n*–Butanol	45%
Acetic acid	10%	Ethanol	5%
Water	50%	Water	49%
		Ammonia	1%
RF found	RF given for glucose	RF found	RF given for glucose
0·18	0·18	0·10	0·105

The sugar spot appeared to correspond to glucose. This was also confirmed by running a chromatogram having unknown sugar spot together with a spot of the authentic d(+)–glucose. After development and subsequent spray, the sugar spot was found to be identical with that of the authentic d(+)–glucose. The presence of this sugar was also confirmed by thin layer chromatography. The sugar spot along with the reference spots of glucose, fructose, xylose and galactose were applied on silica gel G plates buffered with boric acid. The solvent system used was *n*–butanol : acetone : water (4 : 5 : 1). The plate was sprayed with freshly prepared anisaldehyde sulphuric acid reagent (95 ml 95% ethanol, 5 ml concentrated H_2SO_4 and 0·5 ml anisaldehyde) and heated at 90–100°C. The sugar spot was found to correspond with the spot of d(+)–glucose.

(*ii*) *Examination of free amino acids* : About 50 ml of the lead-lake filtrate were centrifuged and then decanted. The clear solution was chromatographed on Whatman No. 1 filter paper by using *n*–butanol : acetic acid : water (4 : 1 : 5 v/v) as solvent and ninhydrin in acetone as spray reagent. The chromatographic analysis revealed the presence of glycine. The presence of glycine was also confirmed by running a chromatogram having spots of the solution together with a spot of authentic sample of glycine.

Examination of the Residue (*A*)

The ethanolic residue left after extraction with water was refluxed with methanol, and the methanol soluble fraction was separated. This was recrystallised from small quantities of methanol for several times when a brown coloured amorphous compound, m.p. 205°C, was obtained. This compound was found to be aromatic and it contained neither N,P,S, nor halogens. It did not contain carboxylic or phenolic groups. It decolourised $KMnO_4$ solution but not bromine water and gave

silver mirror with Tollens reagent. It was insoluble in benzene, ethylacetate, acetone, ether, chloroform but soluble in methanol and ethanol.

Found	Calculated for $C_{10}O_{14}O_3$
C = 66%	C = 65·99%
H = 7·65%	H = 7·7 %
O = 26·35%	O= 26·31%
Mol. wt.	Mol. wt.
172·8 (Semi-micro. Rast)	182

The compound was submitted to I. R. analysis. Peaks appeared at 3400 cm^{-1}, 2940 cm^{-1}, 1650 cm^{-1}, 1450, 1375 cm^{-1} indicating the presence of –OH, $-OCH_3$, substituted phenyl ring and $-\underset{|}{\overset{\overset{\displaystyle CH_3}{|}}{CH}}$ groups.

Further study of the compound is in progress.

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The confluent Hypergeometric functions of three variables

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Abstract

Recently R. N. Jain has defined thirty-eight confluent hypergeometric functions of three variables by confluence from Lauricella and Saran's functions. In this paper five more such functions have been defined and their properties have been studied.

1. Introduction

This paper is devoted to the study of certain new functions which may be regarded as limiting cases of "Hypergeometric functions of three variables" discovered by H. M. Srivastava[8] and R. C. Pandey[9] and thus adding five more functions to the list already discovered by R. N. Jain[7].

There are five new hypergeometric functions of three variables besides fourteen which are already given by Lauricella and Saran. If we use the usual notation $(a, n) = a\,(a+1)\,(a+2)\,.\,.\,.\,.\,.\,(a+n-1)\,;\,(a, 0) = 1$

where a is arbitrary and n a positive integer, then these five functions are as follows :

(1·1) $$H_A\,(\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+p)\,(\beta, m+n)\,(\beta', n+p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, m)\,(\gamma', n+p)}\, x^m y^n z^p,$$

converges absolutely when $|x| < r$, $|y| < s$, $|x| < t$ such that $r+s+t = 1+st$

(1·2) $$H_B\,(\alpha, \beta, \beta'; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+p)\,(\beta, m+n)\,(\beta', n+p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n)\,(\gamma_3, p)}\, x^m y^n z^p,$$

converges absolutely when $|x| < r$, $|y| < s$, $|z| < t$ such that $r+s+t+2\sqrt{rst} = 1$

(1·3) $$H_C\,(\alpha, \beta, \beta'; \gamma\,; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+p)\,(\beta, m+n)\,(\beta', n+p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, m+n+p)}\, x^m y^n z^p,$$

where for convergence, $|x| < 1$, $|y| < 1$, and $|z| < 1$.

(1·4) $$G_A\left(\tfrac{1}{a}, \alpha, \alpha, \beta_1, \beta_2, \beta_1 ; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma, n+p-m)} x^m y^n z^p,$$

where for convergence, $|x| < 1$, $|y| < 1$, $|z| < 1$.

$$G_B\left(\tfrac{1}{a}, \alpha, \alpha, \beta_1, \beta_2, \beta_3, ; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

(1·5) $$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma, n+p-m)} x^m y^n z^p,$$

where for convergence $|x| < 1$, $|y| < 1$, $|z| < 1$.

2. Formation of the confluent Hypergeometric series.

The confluent hypergeometric functions of three variables may be formed by confluence from Srivastava's and Pandey's functions in the following way :

In Srivastava's function H_A defined by (1·1), change y to $\frac{y}{\beta'}$, z to $\frac{z}{\beta'}$ and making $\beta' \to \infty$. We obtain the function

(2·1) $${}_3H_A^{(1)}(\alpha, \beta ; \gamma, \gamma' ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+p)(\beta, m+n)}{(1, m)(1, n)(1, p)(\gamma, m)(\gamma', n+p)} x^m y^n z^p.$$

Taking next the functions (1·2)—(1·5) and applying to them a similar process, we obtain four more new functions, namely

(2·2) $${}_3H_B^{(1)}(\alpha, \beta ; \gamma_1, \gamma_2, \gamma_3 ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+p)(\beta, m+n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n)(\gamma_3, p)} x^m y^n z^p,$$

(2·3) $${}_3G_A^{(1)}\left(\tfrac{1}{a}, \alpha, \alpha, \beta_1, \beta_1 ; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)(\beta_1, m+p)}{(1, m)(1, n)(1, p)(\gamma, n+p-m)} x^m y^n z^p,$$

(2·4) $${}_3G_A^{(2)}\left(\tfrac{1}{a}, \alpha, \alpha, \beta_2 ; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma, n+p-m)} x^m y^n z^p,$$

$$(2\cdot5) \qquad {}_3G_B^{(1)}\left(\frac{1}{a}, a, a, \beta_1, \beta_2 ; \frac{1}{a}, \gamma, \gamma ; x, y, z\right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)(\beta_1, m)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma, n+p-m)} x^m y^n z^p.$$

3. Relations between functions

The five confluents functions which we have defined by triple power series may also be represented by power series in x, y or z as below :

$$(3\cdot1) \qquad {}_3H_A^{(1)}(\alpha, \beta ; \gamma, \gamma'; x, y, z) = \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(1, m)(\gamma, m)} x^m \psi_2(\beta+m, \alpha+m; \gamma'; y, z)$$

$$= \sum_{n=0}^{\infty} \frac{(\beta, n)}{(1, m)(\gamma', n)} y^n \psi_1(\alpha, \beta+n; \gamma, \gamma'+n; x, z)$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha, p)}{(1, p)(\gamma', p)} z^p \psi_1(\beta, \alpha+p; \gamma, \gamma'+p; x, y)$$

Similar formulae can be written for the other confluent functions.

4. Integral representations

(*a*) *Eulerian Type.* From the definition of ${}_3G_A^{(1)}$, we have

$${}_3G_A^{(1)}\left(\frac{1}{a}, \alpha, a, \beta_1, \beta_1; \frac{1}{\gamma}, \gamma, \gamma; x, y, z\right) = \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta_1, m)}{(1, m)(\gamma, -m)} x^m \Phi_1(\alpha-m, \beta_1+m; \gamma-m; y, z)$$

using the integral[6]

$$\Phi_1(\alpha, \beta, \gamma ; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} e^{uy} du$$

$$[Rl(\gamma) > Rl(\alpha) > 0]$$

we find that

$${}_3G_A^{(1)} = \frac{\Gamma(\gamma)}{\Gamma(a)\Gamma(\gamma-a)} \sum_{m=0}^{\infty} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-uy)^{-\beta_1} e^{uz} \frac{(\beta_1, m)}{(1, m)} \left\{\frac{x}{u(1-uy)}\right\}^m du$$

Now if we take $|x| < \rho$, $|y| < \rho'$

$$\left|\frac{x}{u(1-uy)}\right| \leqslant \frac{\rho}{1-\rho'} < 1 \text{ if } \rho+\rho' < 1.$$

Thus the series converges uniformely over the region of integration and order of summation and integration can therefore be reversed and we have

$$(4\cdot1) \qquad {}_3G_A^{(1)} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha+\beta_1-1}(1-u)^{\gamma-\alpha-1}(u-u^2y-x)^{-\beta_1} \cdot e^{uz} du$$

$Rl(\gamma) > R(\alpha) > 0$, $Rl(\alpha+\beta_1) > 0$ and $\rho+\rho' < 1$ ($|x| < \rho$, $|y| < \rho'$).

where for convergence, $|x| < 1$, $|y| < 1$, and $|z| < 1$.

$$(1{\cdot}4) \qquad G_A\left(\tfrac{1}{a}, \alpha, \alpha, \beta_1, \beta_2, \beta_1 ; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)\,(\beta_1, m+p)\,(\beta_2, n)}{(1, m)\,(1, n)(1, p)\,(\gamma, n+p-m)}\, x^m y^n z^p,$$

where for convergence, $|x| < 1$, $|y| < 1$, $|z| < 1$.

$$G_B\left(\tfrac{1}{a}, \alpha, \alpha, \beta_1, \beta_2, \beta_3 ; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

$$(1{\cdot}5) \qquad = \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)\,(\beta_1, m)\,(\beta_2, n)\,(\beta_3, p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, n+p-m)}\, x^m y^n z^p,$$

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$$(2{\cdot}1) \qquad {}_3H_A^{(1)}(\alpha, \beta ; \gamma, \gamma' ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+p)\,(\beta, m+n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, m)\,(\gamma', n+p)}\, x^m y^n z^p.$$

Taking next the functions (1·2)—(1·5) and applying to them a similar process, we obtain four more new functions, namely

$$(2{\cdot}2) \qquad {}_3H_B^{(1)}(\alpha, \beta ; \gamma_1, \gamma_2, \gamma_3 ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+p)\,(\beta, m+n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n)\,(\gamma_3, p)}\, x^m y^n z^p,$$

$$(2{\cdot}3) \qquad {}_3G_A^{(1)}\left(\tfrac{1}{a}, \alpha, \alpha, \beta_1, \beta_1 ; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)\,(\beta_1, m+p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, n+p-m)}\, x^m y^n z^p,$$

$$(2{\cdot}4) \qquad {}_3G_A^{(2)}\left(\tfrac{1}{a}, \alpha, \alpha, \beta_2 ; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)\,(\beta_2, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, n+p-m)}\, x^m y^n z^p,$$

(2·5) $$ {}_3G_B^{(1)}\left(\frac{1}{a},\ a, \alpha, \beta_1, \beta_2;\ \frac{1}{a},\ \gamma, \gamma;\ x, y, z\right) $$

$$ = \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)\,(\beta_1, m)\,(\beta_2, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, n+p-m)}\ x^m y^n z^p. $$

3. Relations between functions

The five confluents functions which we have defined by triple power series may also be represented by power series in x, y or z as below :

(3·1) $$ {}_3H_A^{(1)}(\alpha, \beta;\ \gamma, \gamma';\ x, y, z) = \sum_{m=0}^{\infty} \frac{(\alpha, m)\,(\beta, m)}{(1+m)\,(\gamma, m)}\ x^m\ \Psi_2(\beta+m, \alpha+m;\ \gamma';\ y, z) $$

$$ = \sum_{n=0}^{\infty} \frac{(\beta, n)}{(1, m)\,(\gamma', n)}\ y^n\ \psi_1(\alpha, \beta+n;\ \gamma, \gamma'+n;\ x, z) $$

$$ = \sum_{p=0}^{\infty} \frac{(\alpha, p)}{(1, p)\,(\gamma', p)}\ z^p\ \psi_1(\beta, \alpha+p;\ \gamma, \gamma'+p;\ x, y) $$

Similar formulae can be written for the other confluent functions.

4. Integral representations

(*a*) *Eulerian Type.* From the definition of ${}_3G_A^{(1)}$, we have

$$ {}_3G_A^{(1)}\left(\frac{1}{a},\ \alpha, a, \beta_1, \beta_1;\ \frac{1}{\gamma},\ \gamma, \gamma; x, y, z\right) = \sum_{m=0}^{\infty} \frac{(\alpha, -m)\,(\beta_1, m)}{(1, m)\,(\gamma, -m)}\ x^m\ \Phi_1(\alpha-m, \beta_1+m;\ \gamma-m;\ y, z) $$

using the integral[6]

$$ \Phi_1(\alpha, \beta, \gamma;\ x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}\,(1-u)^{\gamma-\alpha-1}\,(1-ux)^{-\beta}\, e^{uy} du $$

$$ [Rl(\gamma) > Rl(\alpha) > 0] $$

we find that

$$ {}_3G_A^{(1)} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)} \sum_{m=0}^{\infty} \int_0^1 u^{\alpha-1}\,(1-u)^{\gamma-\alpha-1}\,(1-uy)^{-\beta_1}\, e^{uz}.\frac{(\beta_1, m)}{(1, m)} \left\{\frac{x}{u(1-uy)}\right\}^m du $$

Now if we take $|x| < \rho,\ |y| < \rho'$

$$ \left|\frac{x}{u(1-uy)}\right| \leqslant \frac{\rho}{\rho'-\rho} | < | \text{ if } \rho+\rho' < 1. $$

Thus the series converges uniformely over the region of integration and order of summation and integration can therefore be reversed and we have

(4·1) $$ {}_3G_A^{(1)} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha+\beta_1-1}\,(1-u)^{\gamma-\alpha-1}\,(u-u^2y-x)^{-\beta_1}\,.\,e^{uz}\,du $$

$Rl(\gamma) > R(\alpha) > 0,\ Rl(\alpha+\beta_1) > 0$ and $\rho+\rho' < 1\ (|x| < \rho,\ |y| < \rho')$.

Similarly we can prove

(4·2) $${}_3G_A^{(2)}\left(\tfrac{1}{\alpha}, \alpha, \alpha, \beta_2 ; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}\,(1-u)^{\gamma-\alpha-1}(1-uy)^{-\beta_2}\,e^{uz+x/u}\,du$$

where $Rl(\gamma) > R(\alpha) > 0,\ |x| < 1.$

(4·3) $${}_3G_B^{(1)}\left(\tfrac{1}{\alpha}, \alpha, \alpha, \beta_1, \beta_2; \tfrac{1}{\gamma}, \gamma, \gamma ; x, y, z\right)$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha+\beta_1-1}(1-u)^{\gamma-\alpha-1}\,(1-uy)^{-\beta_2}(u-x)^{-\beta_1}\,e^{uz}du$$

where $Rl(\gamma) > R(\alpha) > 0,\ Rl(\alpha+\beta_1) > 0,\ |x| < 1.$

(*b*) *Pochhammar's double loop type.* From the definition of ${}_3H_A^{(1)}$, we get

$${}_3H_A^{(1)}(\alpha, \beta ; \gamma, \gamma'; x, y, z) = \sum_{p=0}^{\infty}\frac{(\alpha, p)}{(1,p)\,(\gamma',p)}\,z^p\,\psi_1(\beta, \alpha+p ; \gamma, \gamma'+p ; x, y)$$

using the integral

$$\psi_1(\rho+\rho_1-1,\beta; \gamma,\gamma_1; x,y) = \frac{\Gamma(\rho)\,\Gamma(\rho_1)\,\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\int (-t)^{-\rho}(t-1)^{-\rho_1}\,{}_2F_1(\rho,\beta;\gamma; \tfrac{x}{t}) \times$$
$$\times\ {}_1F_1(\rho_1; \gamma_1 ; \tfrac{y}{1-t})\,dt$$

where $|t| > |x|,\ |1-t| > |y|$ along the contour. [The contour of integration is a Pochhammar's double loop type $(1+, 0+, 1-, 0-)$ and t^ρ, etc ; have their principal values], we get

$${}_3H_A^{(1)} = \frac{\Gamma(\rho)\,\Gamma(\rho_1)\,\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\int (-t)^{-\rho}(t-1)^{-\rho_1}\,{}_2F_1(\rho, \alpha+p; \gamma; \tfrac{x}{t}).\ {}_1F_1(\rho_1; \gamma'+p; \tfrac{y}{1-t}) \times$$
$$\times \sum_{p=0}^{\infty}\frac{(\alpha, p)}{(1, p)\,(\gamma', p)}\,z^p.\,dt$$

Since the series for p is absolutely convergent for $|t| > |x|,\ |1-t| > |y|,\ |z| < 1$, the change in the order of integration and summation is permissible and we have

(4·4) $${}_3H_A^{(1)} = \frac{\Gamma(\rho)\Gamma(\rho_1)\,\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\int (-t)^{-\rho}(t-1)^{-\rho_1}\,{}_3\Phi_P^{(1)}(\rho,\rho_1, \alpha,\alpha; \gamma,\gamma',\gamma'; \tfrac{x}{t}, \tfrac{y}{1-t}, z)\,dt$$

where $|t| > |x|,\ |1-t| > |y|,\ |z| < 1,\ \beta = \rho+\rho_1-1$ along the contour and for definition of ${}_3\Phi_P{}^{(2)}$, see[7].

Similarly, we obtain,

(4·5) $${}_3H_B^{(1)}(\alpha, \beta ; \gamma_1\,\gamma_2, \gamma_3 ; x, y, z)$$

$$=\frac{\Gamma(\rho)\,\Gamma(\rho_1)\,\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\int (-t)^{-\rho}(t-1)^{-\rho_1}\,{}_1F_1\left(\rho_1 ; \gamma_2; \tfrac{y}{1-t}\right)\psi\left(\alpha,\rho;\gamma_1,\gamma_3; \tfrac{x}{t}, z\right)dt$$

where $|t| > |x|$, $|1-t| > |y|$, $|z| < 1$, $\beta = \rho+\rho_1-1$, along the contour.

(c) *Mellin–Barne's type.* From the definition of ${}_3H_B^{(1)}$, we have

$${}_3H_B^{(1)}(\alpha, \beta; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \sum_{p=0}^{\infty} \frac{(\alpha, p)}{(1, p)(\gamma_3, p)} \psi_1(\beta, \alpha+p; \gamma_1, \gamma_2; x, y)\, z^p$$

using the integral

$$\psi_1(\alpha, \beta; \gamma, \gamma_1; x, y) = \frac{\Gamma(\gamma_1)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} {}_1F_1(\alpha+t; \gamma; x). \frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)} \Gamma(-t)(-y)^t\, dt$$

$${}_3H_B^{(1)} = \frac{\Gamma(\gamma_2)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{1}{2\pi i} \sum_{p=0}^{\infty} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\beta+t)\Gamma(\alpha+t)}{\Gamma(\gamma_2+t)} \Gamma(-t)(-y)^t\, {}_1F_1(\beta+t; \gamma_1; x) \times$$

$$\times \frac{(\alpha+t, p)}{(1, p)(\gamma_3, p)} z^p.\, dt$$

Changing the order of integration and summation which is easily justifiable for $|x| < 1$, $|y| < 1$, we get

$$= \frac{\Gamma(\gamma_2)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma_2+t)} \Gamma(-t)(-y)^t.\ {}_1F_1(\alpha+t; \gamma_3; z).\ {}_1F_1(\beta+t; \gamma_1; x)\, dt \tag{4·6}$$

Again using the integral for ${}_1F_1$[3] we can prove

$$= \frac{\Gamma(\gamma_2)\Gamma(\gamma_3)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha+t+s)\Gamma(\beta+t)}{\Gamma(\gamma_2+t)\Gamma(\gamma_3+s)} \Gamma(-s)\Gamma(-t)(-y)^t(-z)^s \times\ {}_1F_1(\beta+t; \gamma_1; x)\, ds\, dt \tag{4·7}$$

$$= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{1}{(2\pi i)^3} \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha+t+s)\Gamma(\beta+t+r)}{\Gamma(\gamma_1+r)\Gamma(\gamma_2+s)\Gamma(\gamma_3+t)} \times \Gamma(-r)\Gamma(-s)\Gamma(-t)(-x)^r(-y)^s(-z)^t\, dr\, ds\, dt. \tag{4·8}$$

Proceeding in the same manner, we obtain

$${}_3H_A^{(1)}(\alpha, \beta; \gamma, \gamma'; x, y, z). \tag{4·9}$$

$$= \frac{\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)} \Gamma(-t)(-y)^t.\ \Phi_1(\beta+t, \alpha+t; \gamma'; x, z)\, dt.$$

5. Transformations

We can easily write

$${}_3H_A^{(1)}(\alpha, \beta; \gamma, \gamma'; x, y, z) = \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(1, m)(\gamma, m)} \Phi_2(\beta+m, \alpha+m; \gamma'; y, z)\, x^m$$

Using the formula[1]

$$\Phi_2(\beta, \beta'; \gamma; x, y) = (1-x)^{-\beta}(1-y)^{-\beta'}\, \Phi_2\left(\beta+m, \alpha<m; \gamma'; \frac{x}{x-1}, \frac{y}{y-1}\right)$$

we get the identity.

$$(5\cdot1)\qquad {}_3H_A^{(1)}(\alpha,\beta;\gamma,\gamma';x,y,z)=(1-y)^{-\beta}(1-z)^{-\alpha}\,{}_3H_A^{(1)}\left(\alpha,\beta;\gamma,\gamma';\frac{x}{(1-y)(1-z)},\frac{y}{y-1},\frac{z}{z-1}\right)$$

and if we employ the formula

$$\Phi_2(\beta,\beta';\gamma;x;y)=(1-x)^{-\beta}\,\Phi_2\left(\beta,\beta';\gamma;\frac{x}{x-1},y\right)$$

we get

$$(5\cdot2)\qquad {}_3H_A^{(1)}(\alpha,\beta;\gamma,\gamma';x,y,z)=(1-y)^{-\beta}\,{}_3H_A^{(1)}\left(\alpha,\beta;\gamma,\gamma';\frac{x}{1-y},\frac{y}{y-1},z\right).$$

A similar transformation follows if we interchange y and z. Proceeding in the same manner, we obtain

$$(5\cdot3)\qquad {}_3G_A^{(1)}\left(\frac{1}{a},\alpha,\alpha,\beta_1\,\beta_1;\frac{1}{\gamma},\gamma,\gamma;x,y,z\right)$$

$$=(1-z)^{-\alpha}\,{}_3G_A^{(1)}\left(\frac{1}{a},\alpha,\alpha,\beta_1,\beta_1;\frac{1}{\gamma},\gamma,\gamma;x(1-z),\frac{z-y}{z-1},\frac{z}{z-1}\right)$$

$$(5\cdot4)\qquad {}_3G_A^{(2)}\left(\frac{1}{a},\alpha,\alpha,\beta_2;\frac{1}{\gamma},\gamma,\gamma;x,y,z\right)$$

$$=(1-z)^{-\alpha}\,{}_3G_A^{(2)}\left(\frac{1}{a},\alpha,\alpha,\beta_2;\frac{1}{\gamma},\gamma,\gamma;x(1-z),\frac{z-y}{z-1},\frac{z}{z-1}\right)$$

Again on putting $u-u^2y-x=-x(1-us)(1-ut)$

where $$s+t=\frac{1}{x}\text{ and }st=\frac{y}{x}$$

in the integral (4·1), we get

$$(5\cdot5)\qquad {}_3G_A^{(1)}\left(\frac{1}{a},\alpha,\alpha,\beta_1,\beta_1;\frac{1}{\gamma},\gamma,\gamma;x,y,z\right)$$

$$=\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\gamma+\beta)}\cdot(-x)^{-\beta_1}\,{}_3\Phi_D^{(1)}(\alpha+\beta_1,\alpha+\beta_1,\alpha+\beta_1,\beta_1,\beta_1;\gamma+\beta_1,\gamma+\beta_1,\gamma+\beta_1;s,z,t)$$

a relation between ${}_3G_A^{(1)}$ and ${}_3\Phi_D^{(1)}$. For the definition of ${}_3\Phi_D^{(1)}$, see[7].

6. Now we derive an interesting integral for the function ${}_3G_B^{(1)}$.

Since $${}_3G_B^{(1)}\left(\frac{1}{a},\alpha,\alpha,\beta_1,\beta_2;\frac{1}{\gamma},\gamma,\gamma;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(\alpha,p)}{(1;p)(\gamma,p)}G_2(\beta_1,\beta_2,\alpha+p,1-\gamma-p;-x,-y)\,z^p$$

where G_2 is Horn's functions[3].

Using the known integral[5]

$$G_2(\alpha, \alpha', \beta, \beta'; x, y) = \frac{\Gamma(1-\beta)\,\Gamma(1-\beta')}{2i\pi\,\Gamma(1-\beta-\beta')} \int_{-1}^{(o,x+)} u^{\beta-1}\,(u+1)^{-\beta-\beta'}\,(1-\tfrac{x}{u})^{-\alpha}(1-uy)^{-\alpha'}\,du$$

$$[Rl(\beta+\beta') < 1]$$

we have

$$ {}_3G_B^{(1)} \frac{\Gamma(\gamma)\,\Gamma(1-\alpha)}{2\pi i\,\Gamma(\gamma-\alpha)} \sum_{p=0}^{\infty} \int_{-1}^{(o,x+)} u^{\alpha-1}\,(1+u)^{\gamma-\alpha-1}\,(1+\tfrac{x}{u})^{-\beta_1}(1+uy)^{-\beta_2} \times \frac{1}{(1,p)}\,(-uz)^p\,du$$

The change of order of integeration and summation is easily justifible for $|z| < 1$, and therefore, we have

$$(6\cdot1) \qquad = \frac{\Gamma(\gamma)\,\Gamma(1-\alpha)}{2\pi i\,\Gamma(\gamma-\alpha)} \int_{-1}^{(o,x+)} u^{\alpha-1}\,(1+u)^{\gamma-\alpha-1}\,(1+\tfrac{x}{u})^{-\beta_1}(1+uy)^{-\beta_2}\,e^{-uz}.\,du$$

Again on making the substitution $u = \dfrac{v}{1-y-vy}$, in the above integral, we are immediatly led to the transformation

$$(6\cdot2) \qquad {}_3G_A^{(1)}\left(\tfrac{1}{\alpha}, \alpha, \alpha, \beta_1; \tfrac{1}{\gamma}, \gamma, \gamma; x, y, z\right) = (1-y)^{-\alpha}\,(1-xy)^{-\beta_1}$$

$$ {}_3G_A^{(1)}\left(\frac{1}{\alpha}, \alpha, \alpha, \beta_1; \frac{1}{\gamma}, \gamma, \gamma; \frac{x(1-y)}{1-xy}, \frac{-y}{1-y}, \frac{z-y}{1-y}\right)$$

and when $y = z$,

$$(6\cdot3) \qquad {}_3G_B^{(1)}\left(\tfrac{1}{\alpha}, \alpha, \alpha, \beta_1, \beta_2; \tfrac{1}{\gamma}, \gamma, \gamma; x, y, y\right) = (1-y)^{-\alpha}(1-xy)^{-\beta_1}$$

$$\Gamma_1\left(\beta_1, \alpha, 1-\gamma; \frac{-x(1-y)}{1-xy}, \frac{y}{1-y}\right)$$

where Γ_1 is Horn's function[3].

7. Expansions

If we introduce the inverse pair of symbolic operators

$$\nabla_{x,y}(h) = \frac{\Gamma(h)\,\Gamma(\delta+\delta'+h)}{\Gamma(\delta+h)\,\Gamma(\delta'+h)}$$

$$\text{and } \triangle_{x,y}(h) = \frac{\Gamma(\delta+h)\,\Gamma(\delta'+h)}{\Gamma(h)\,\Gamma(\delta+\delta'+h)}$$

where δ and δ' stand for $x\dfrac{\partial}{\partial x}$ and $y\dfrac{\partial}{\partial y}$ respectively, we can easily show that

$$(7\cdot1) \qquad {}_3H_A^{(1)}(\alpha, \beta; \gamma, \gamma'; x, y, z) = \nabla_{x,y}(\beta)\,{}_3\Phi_M^{(2)}(\beta, \alpha, \beta, \beta; \gamma, \gamma', \gamma'; x, y, z)$$

$$(7\cdot2) \qquad {}_3\Phi_M^{(2)}(\beta, \alpha, \beta, \beta; \gamma, \gamma', \gamma'; x, y, z) = \triangle_{x,y}(\beta)\,{}_3H_A^{(1)}(\alpha, \beta; \gamma, \gamma'; x, y, z)$$

where $\qquad {}_3\Phi^{(2)}_M$ is Jain's function[7]

These operational relations lead to the following expansions

(7·3) $\quad {}_3H^{(1)}_A(\alpha, \beta ; \gamma, \gamma'; x, y, z) = \sum_{r=0}^{\infty} \frac{(\alpha,r)\,(\beta, r)}{(1, r)\,(\gamma, r)\,(\gamma', r)} x^r y^r \times$

$$\times\ {}_3\Phi^{(2)}_M(\beta+r, \alpha+r, \beta+r, \alpha+r ; \gamma+r, \gamma'+r, \gamma'+r; x, y, z)$$

(7·4) $\quad {}_3\Phi^{(2)}_M(\beta, \alpha, \beta, \beta; \gamma, \gamma', \gamma'; x, y, z) = \sum_{r=0}^{\infty} \frac{(-1)^r\,(\alpha, r)\,(\beta, r)}{(1, r)\,(\gamma, r)\,(\gamma', r)} x^r y^r \times$

$$\times\ {}_3H^{(1)}_A(\alpha+r, \beta+r ; \gamma+r, \gamma'+r ; x, y, z)$$

8. Differential Equations Satisfied by the Functions

The five confluent functions satisfy partial differential equations of rather simple forms, which it is easy to obtain, by confluence from the systems of equations found by Srivastava[8] and Bhatt[2] for the H and G functions respectively.

Thus we find that the system for the function ${}_3H^{(1)}_B$ is :

(8·1)
$$\begin{cases} [\theta(\theta+\gamma_1-1) - x\,(\theta+\psi+\alpha)\,(\theta+\phi+\beta)]\ {}_3H^{(1)}_B = 0 \\ [\Phi(\phi+\gamma_2-1) - y\,(\theta+\phi+\beta)\,(\phi+\psi)]\ {}_3H^{(1)}_B = 0 \\ [\psi(\psi+\gamma_3-1) - z\,(\theta+\psi+\alpha)\,(\phi+\psi)]\ {}_3H^{(1)}_B = 0 \end{cases}$$

where $\qquad \theta = x\frac{\partial}{\partial x},\ \Phi = y\frac{\partial}{\partial y}$ and $\psi = z\frac{\partial}{\partial z}$.

and similarly for other functions.

We now illustrate the classical method of solution in series by integerating the differential system (8·1) in the neighbourhood of origin.

Let us assume a solution of (8·1) in the form

$${}_3H^{(1)}_B = x^g y^h z^k \sum_{m,n,p=0}^{\infty} A_{m,n,p}\, x^m y^n z^p,$$

where g, h, k are suitable constants.

The indicial equations of the system are given by

$$\begin{aligned} g(g+\gamma_1-1) &= 0 \\ h(h+\gamma_2-1) &= 0 \\ k(k+\gamma_3-1) &= 0 \end{aligned} \qquad \text{These gives} \qquad \begin{aligned} g &= 0 \text{ or } 1-\gamma_1 \\ h &= 0 \text{ or } 1-\gamma_2 \\ k &= 0 \text{ or } 1-\gamma_3 \end{aligned}$$

The above roots of the indicial equation lead to the following eight possible sets of values of the parameters g, h, k.

$g=0$	0	0	$1-\gamma_1$	$1-\gamma_1$	0	$1-\gamma_1$	$1-\gamma_1$
$h=0$	$1-\gamma_2$	0	0	$1-\gamma_2$	$1-\gamma_2$	0	$1-\gamma_2$
$k=0$	0	$1-\gamma_3$	0	0	$1-\gamma_3$	$1-\gamma_3$	$1-\gamma_3$

These lead to the following general solution of (8·1) valid in the neighbourhood of the origin.

$$
\begin{aligned}
{}_3H_B^{(1)} &= A\,{}_3H_A^{(1)}(\alpha,\beta;\gamma_1\gamma_2,\gamma_3;x,y,z)\\
&+B_1\,x^{1-\gamma_1}\,{}_3H_B^{(1)}(1-\gamma_1+\alpha,1-\gamma_1+\beta;2-\gamma_1,\gamma_2,\gamma_3;x,y,z)\\
&+B_2\,y^{1-\gamma_2}\,{}_3H_B^{(1)}(\alpha,1-\gamma_2+\beta;\gamma_1,2-\gamma_2,\gamma_3;x,y,z)\\
&+B_3\,z^{1-\gamma_3}\,{}_3H_B^{(1)}(1-\gamma_3+\alpha,\beta;\gamma_1,\gamma_2,2-\gamma_3;x,y,z)\\
&+C_1\,x^{1-\gamma_1}y^{1-\gamma_2}\,{}_3H_B^{(1)}(1-\gamma_1+\alpha,2-\gamma_1-\gamma_2+\beta;2-\gamma_1,2-\gamma_2,\gamma_3;x,y,z)\\
&+C_2\,y^{1-\gamma_2}z^{1-\gamma_3}\,{}_3H_B^{(1)}(1-\gamma_3+\alpha,1-\gamma_2+\beta;\gamma_1,2-\gamma_2,2-\gamma_3;x,y,z)\\
&+C_3\,x^{1-\gamma_1}z^{1-\gamma_3}\,{}_3H_B^{(1)}(2-\gamma_3-\gamma_1+\alpha,1-\gamma_1+\beta;2-\gamma_1,\gamma_2,2-\gamma_3;x,y,z)\\
&+D\,x^{1-\gamma_1}y^{1-\gamma_2}z^{1-\gamma_3}\,{}_3H_B^{(1)}(2-\gamma_3-\gamma_1+\alpha,2-\gamma_1-\gamma_2+\beta;2-\gamma_1,2-\gamma_2,2-\gamma_3;x,y,z)
\end{aligned}
$$

where A, B's, C's and D are arbitrary and constants.

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Rayleigh's Wave in a Thermoelastic Medium with Sinusoidal Wavy Boundary

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Abstract

Rayleigh's Wave in a thermoelastic medium in sinusoidal wavy boundary has been considered. In solving the problem perturbation method has been used and ε^2 and its higher order have been neglected. Three types of surface wave have been found, one of them is the same as Rayleigh wave in thermoelastic medium with plane boundary and the other two are of different nature depending on the wave length and the wavy nature of the boundary.

Introduction

In plane boundary Lockett[1] has considered Rayleigh's wave in a thermoelastic medium and has shown that the difference between the velocities of propagation of Rayleigh's wave in thermoelastic medium is less than that in a medium where thermal effect is not considered by one percent. In this paper we shall consider the more practical case when the thermoelastic medium is a semi-infinite space with a sinusoidal wavy boundary of the form $y = \varepsilon f(x) = h \sin rx$ where $\varepsilon = \frac{hr}{\pi} << 1$. In solving this problem perturbation method due to Eringen[2] has been used and ε^2 and its higher order have been neglected. The wavy boundary has a normal traction of the concentrated type and zero shear. We have assumed a convection condition for the temperature on the boundary. The particular case $h = 0$ deals with thermal insulation on the boundary. In the case $h = 0$

$$\beta^2 = \frac{\lambda + 2\mu}{\mu} = 3$$

(*i. e.* for $\lambda = \mu$) as assumed by Lockett, the expression for the displacements in the wavy boundary have been obtained and found, of course neglecting body waves, that the surface waves break into three parts one of them is the same as Rayleigh's wave in thermoelastic medium with plane boundary and the other two are of different nature depending on the wave length and the wavy nature of the boundary.

Formulation of the problem and its solution

The starting point of our consideration is the system of wave equations

$$\left(\nabla_1^2 - \frac{1}{\chi}\partial_t\right)\left(\nabla_1^2 - \frac{1}{c_1^2}\partial t^2\right)\Phi - \frac{\varepsilon'}{\chi}\partial t\, \nabla_1^2\Phi = 0 \left(\nabla_1^2 - \frac{1}{c_2^2}\partial t^2\right)\Psi = 0 \qquad (1)$$

Where $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\partial_t = \frac{\partial}{\partial t}$, ε' = coupling constant and other symbol has usual significance. The temperature equation is given by

$$T = \frac{1}{m}\left(\nabla_1^2 - c_1^{-2}\,\partial_t^2\right)\Phi \tag{2}$$

The equation of the sinusoidal wavy boundary is taken as

$$y = \varepsilon f(x) = h \sin r x \text{ and } \varepsilon = \frac{hr}{\pi} < < 1,$$

represents a small perturbation parameter.

The boundary condition at $y = \varepsilon f(x)$ read as follows

$$\left.\begin{aligned} \sigma_n &= \sigma_{xx}\, n_x^2 + 2\,\tau_{xy}\, n_x\, n_y + \sigma_{yy}\, n_y^2 \\ \tau_n &= (\sigma_{xx} - \sigma_{yy})\, n_x\, n_y + \tau_{xy}\,(n_y^2 - n_x^2) \\ \frac{\partial T}{\partial n} + h\,T &= n_x\, T_x + n_y\, T_y + hT \end{aligned}\right\} \tag{3}$$

where σ_n, τ_n are prescribed functions at the boundary curve, and n_x and n_y are the components of the inner unit normal which are given by

$$\left.\begin{aligned} n_x &= -\,\varepsilon f'\,(1+\varepsilon^2 f'^2)^{-\frac{1}{2}} \\ n_y &= (1+\varepsilon^2 f'^2)^{-\frac{1}{2}} \end{aligned}\right\} \tag{4}$$

where prime represent differentiation with respect to x. Substituting (4) in (3) and expanding in power series in ε and neglecting ε^2 and higher orders the boundary conditions with normal traction and zero shear reads

$$\left.\begin{aligned} \sigma_n &= \sigma_{yy} - 2\,\varepsilon f'\,\tau_{xy} = \sigma(s) \\ \tau_n &= \tau_{xy} + \varepsilon f'\,(\sigma_{yy} - \sigma_{xx}) = 0 \\ \frac{\partial T}{\partial y} &- \varepsilon f' \frac{\partial T}{\partial x} + h\,T = 0 \end{aligned}\right\} \tag{5}$$

where S is the arc length along the boundary curve and for S

$$\left.\begin{aligned} S &= \int_0^x (1+\varepsilon^2 f'^2)^{\frac{1}{2}}\, dx = x + 0\,(\varepsilon^2) \\ \therefore\ \sigma_n &= \sigma\,(s) = \sigma\,(x) + 0(\varepsilon^2). \end{aligned}\right\} \tag{5a}$$

If the load is a concentrated type $\sigma\,(x) = P\,\delta\,(x)$ (5b)

where P is the magnitude of the normal force component.

To solve the system of equations in (1) we assume

$$\Phi = e^{i\omega t}\,\phi_0 \qquad \Psi = e^{i\omega t}\,\psi_0$$

and substituting in (1)

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{i\omega}{\chi}\right]\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c_1^2}\right]\phi_0 - \frac{\varepsilon'}{\chi}\, i\omega \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\phi_0 = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c_2^2}\right]\psi_0 = 0 \tag{6}$$

To find the solution of (6) we introduce one dimensional Fourier's transform

$$\left.\begin{aligned} \bar{\phi}_0(\xi y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_0\,(xy)\, e^{i\,\xi\,x} dx \\ \bar{\psi}_0(\xi y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0\,(xy)\, e^{i\,\xi\,x}\, dx \end{aligned}\right\} \tag{7}$$

the inverse of (7) is given by

$$\left.\begin{aligned} \phi_0(xy) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\phi}_0\,(\xi y)\, e^{-i\,\xi\,x}\, d\xi \\ \psi_0(xy) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\psi}_0\,(\xi y)\, e^{-i\,\xi\,x}\, d\xi \end{aligned}\right\} \tag{8}$$

Applying (8) in (6)

$$\left\{ \left[-\xi^2 + \frac{\partial^2}{\partial y^2} - \frac{i\omega}{\chi}\right]\left[-\xi^2 + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c_1^2}\right] - \frac{\varepsilon'}{\chi}\, i\omega \left[-\xi^2 + \frac{\partial^2}{\partial y^2}\right] \right\} \bar{\phi}_0 = 0$$

$$\left[-\xi^2 + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c_2^2}\right] \bar{\psi}_0 = 0$$

or

$$\left.\begin{aligned} &[\,(D^2 - \xi^2 - q)\,(D^2 - \xi^2 + \sigma^2) - q\,\varepsilon'\,(D^2 - \xi^2)\,]\,\bar{\phi}_0\,(\xi y) = 0 \\ &[D^2 - \xi^2 + \tau^2]\,\bar{\psi}_0\,(\xi y) = 0 \end{aligned}\right\} \tag{9}$$

where $\quad q = \frac{i\omega}{\chi}\,;\ \sigma^2 = \frac{\omega^2}{c_1^2}\,;\ \tau^2 = \frac{\omega^2}{c_2^2}\,;\ \xi^2 = \frac{\omega^2}{c^2}\ \ D = \frac{d}{dy}$

The solutions of the system of equations (9) are

$$\left.\begin{aligned} \bar{\phi}_0\,(\xi y) &= A\, e^{-\lambda_1 y} + B\, e^{-\lambda_2 y} \\ \bar{\psi}_0\,(\xi y) &= c\, e^{-\nu y} \end{aligned}\right\} \tag{10}$$

where λ_1, λ_2 are the roots of $(\lambda^2 - \xi^2)^2 + (\lambda^2 - \xi^2)\,(\sigma^2 - q_1) - \sigma^2 q = 0$ and

$$\nu = (\xi^2 - \tau^2)^{\frac{1}{2}},\ q_1 = q\,(1 + \varepsilon')$$

knowing the functions Φ, Ψ we calculate the displacements and stresses and the expression for them are

$$u = u_0 e^{i\omega t} = \left(\frac{\partial\phi_0}{\partial x} - \frac{\partial\psi_0}{\partial y}\right) e^{i\omega t},\ v = v_0 e^{i\omega t} = \left(\frac{\partial\phi_0}{\partial y} + \frac{\partial\psi_0}{\partial x}\right) e^{i\omega t}$$

$$\sigma_{xx} = -2\mu\left[\frac{\partial^2\psi_0}{\partial x\,\partial y} + \frac{\partial^2\phi_0}{\partial y^2}\right] - \rho\,\omega^2\,\phi_0$$

$$\sigma_{yy} = 2\mu\left[\frac{\partial^2\psi_0}{\partial x\,\partial y} - \frac{\partial^2\phi_0}{\partial x^2}\right] - \rho\,\omega^2\,\phi_0$$

$$\tau_{xy} = 2\mu\frac{\partial^2\phi_0}{\partial x\,\partial y} + \mu\left(\frac{\partial^2\psi_0}{\partial x^2} - \frac{\partial^2\psi_0}{\partial y^2}\right).$$

Therefore the displacements and stresses obtained by using (8) and (10) gives

$$u_0 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[-i\xi\left(A e^{-\lambda_1 y}+B e^{-\lambda_2 y}\right)+C\nu e^{-\nu y}\right]e^{-i\xi x}\,d\xi \tag{11}$$

$$v_0 = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[A\lambda_1 e^{-\lambda_1 y}+B\lambda_2 e^{-\lambda_2 y}+i\xi C e^{-\nu y}\right]e^{-i\xi x}\,d\xi \tag{12}$$

$$\sigma_{xx} = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[A a_1 e^{-\lambda_1 y}+B b_1 e^{-\lambda_2 y}+C c_1 e^{-\nu y}\right]e^{-i\xi x}\,d\xi \tag{13}$$

$$\sigma_{yy} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[A a_2 e^{-\lambda_1 y}+B b_2 e^{-\lambda_2 y}+C c_2 e^{-\nu y}\right]e^{-i\xi x}\,d\xi \tag{14}$$

$$\tau_{xy} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[A a_3 e^{-\lambda_1 y}+B b_3 e^{-\lambda_2 y}+C c_3 e^{-\nu y}\right]e^{-i\xi x}\,d\xi \tag{15}$$

where

$$\left.\begin{aligned} a_1 &= \lambda_1^2+\tfrac{1}{2}\tau^2; & b_1 &= \lambda_2^2+\tfrac{1}{2}\tau^2; & c_1 &= i\xi\nu \\ a_2 &= \xi^2-\tfrac{1}{2}\tau^2; & b_2 &= \xi^2-\tfrac{1}{2}\tau^2; & c_2 &= i\xi\nu = c_1 \\ a_3 &= i\xi\lambda_1; & b_3 &= i\xi\lambda_2; & c_3 &= -\tfrac{1}{2}(2\xi^2-\tau^2) \end{aligned}\right\} \tag{16}$$

The boundary conditions (5) with the expressions from (13), (14) and (15) takes the from on $y = \varepsilon f(x)$

$$\frac{\sigma_n}{2\mu} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big\{[a_2-2\varepsilon f' a_3]A e^{-\varepsilon f\lambda_1}+[b_2-2\varepsilon f' b_3]B e^{-\varepsilon f\lambda_2}$$
$$+[c_2-2\varepsilon f' c_3]C e^{-\varepsilon f\nu}\Big\}e^{-i\xi x}\,d\xi \tag{17}$$

$$\frac{\tau_n}{2\mu} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big\{[a_3+\varepsilon f'(a_1+a_2)]A e^{-\varepsilon f\lambda_1}+[b_3+\varepsilon f'(b_1+b_2)]B e^{-\varepsilon f\lambda_2}$$
$$+[c_3+\varepsilon f'(c_1+c_2)]C e^{-\varepsilon f\nu}\Big\}e^{-i\xi x}\,d\xi \tag{18}$$

$$\left(\frac{\partial T}{\partial y}-\varepsilon f'\frac{\partial T}{\partial x}+hT\right)_{on\ y=\varepsilon f} = 0 \text{ gives}$$

$$-\frac{1}{m\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big\{[n_1(\lambda_1-h)-\varepsilon f' i\xi n_1]A e^{-\varepsilon f\lambda_1}$$
$$+[n_2(\lambda_2-h)-\varepsilon f' i\xi n_2]B e^{-\varepsilon f\lambda_2}\Big\}e^{-i\xi x}\,d\xi = 0 \tag{19}$$

where

$$n_{1,2} = \lambda^2_{1,2}+\sigma^2-\xi^2$$

we now write

$$\sigma_n = \sigma_0+\varepsilon\sigma_1+0(\varepsilon^2)$$
$$\tau_n = \tau_0+\varepsilon\tau_1+0(\varepsilon^2)$$
$$A = A_0+\varepsilon A_1+0(\varepsilon^2)$$
$$e^{-\varepsilon f\lambda_1} = 1-\varepsilon f\lambda_1+0(\varepsilon^2)$$

and similar expression for others.

Substituting the above expressions in (17), (18) and (19) and collecting independent term in ε from both sides the following equations are obtained for the determination of A_0, B_0 and C_0

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(a_2 A_0+b_2 B_0+c_2 C_0)\, e^{-i\xi x}\, d\xi=\frac{\sigma_0}{2\mu}$$

$$\left.\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(a_3 A_0+b_3 B_0+c_3 C_0)\, e^{-i\xi x}\, d\xi=\frac{\tau_0}{2\mu}\\ &\int_{-\infty}^{\infty}\{A_0[n_1(\lambda_1-h)]+B_0[n_2(\lambda_2-h)]\}e^{-i\xi x}d\xi=0\end{aligned}\right\} \qquad (20)$$

Taking Fourier transform of the equation (20)

$$\left.\begin{aligned}&a_2 A_0+b_2 B_0+c_2 C_0=\frac{\bar{\sigma}_0}{2\mu}\\ &a_3 A_0+b_3 B_0+c_3 C_0=\frac{\bar{\tau}_0}{2\mu}\\ &A_0[n_1(\lambda_1-h)]+B_0[n_2(\lambda_2-h)]=0\end{aligned}\right\} \qquad (21)$$

Again from (5b)

$$\left.\begin{aligned}&\bar{\sigma}_0=\frac{P}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)\, e^{i\xi x}\, dx=\frac{P}{\sqrt{2\pi}}\\ &\bar{\tau}_0=0\,; \qquad \bar{\sigma}_1=0\,; \qquad \bar{\tau}_1=0\end{aligned}\right\} \qquad (22)$$

The solution of (21) with the help of (22) are

$$\left.\begin{aligned}A_0&=\frac{-n_2(\lambda_2-h)\, c_3}{2\mu D}\cdot\frac{P}{\sqrt{2\pi}}\equiv\frac{\alpha_0(\xi)}{D(\xi)}\\ B_0&=\frac{n_1(\lambda_1-h)\, c_3}{2\mu D}\quad\frac{P}{\sqrt{2\pi}}\equiv\frac{\beta_0(\xi)}{D(\xi)}\end{aligned}\right\} \qquad (23)$$

$$C_0=-\frac{1}{2\mu D}\left\{n_1(\lambda_1-h)\, b_3-n_2(\lambda_2-h)\, a_3\right\}\frac{P}{\sqrt{2\pi}}\equiv\frac{\gamma_0(\xi)}{D(\xi)}\ say$$

where

$$D(\xi)=(2\xi^2-\tau^2)^2-4\xi^2(\xi^2-\tau^2)^{\frac{1}{2}}\ \frac{(\lambda_1 n_2-\lambda_2 n_1)\, h+\lambda_1\lambda_2(n_1-n_2)}{(n_2-n_1)\, h+(n_1\lambda_1-n_2\lambda_2)}. \qquad (24)$$

Again to determine A_1, B_1, C_1 the following sets of equations are obtained by equating the coefficient of ε from both sides of (17), (18) and (19)

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{a_2 A_1+b_2 B_1+c_2 C_1\}\, e^{-i\xi x}\, d\xi=\frac{\sigma_1}{2\mu}+\\ &+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[A_0(a_2 f\lambda_1+2f'\, a_3)+B_0(b_2 f\lambda_2+2f'\, b_3)\\ &\qquad +C_0(c_2 f\nu+2f c_3)]\, e^{-i\xi x}d\xi, \qquad (25)\\ &\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{a_3 A_1+b_3 B_1+c_3 C_1\}\, e^{-i\xi x}\, d\xi\end{aligned}$$

$$= \frac{\tau_1}{2\mu} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\, A_0 \{a_3 f \lambda_1 - f' (a_1 + a_2)\} +$$

and

$$+ B_0 \{ b_3 f \lambda_2 - f' (b_1 + b_2) \} + C_0 \{ c_3 f \nu - f' (c_1 + c_2) \} \,]\, e^{-i \xi x} d\xi \tag{26}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{ A_1 [n_1 (\lambda_1 - h)] + B_1 [n_2 (\lambda_2 - h)] \} e^{-i \xi x} d\xi$$

$$= \int_{-\infty}^{\infty} \{ A_0 [n_1 (\lambda_1 - h) f \lambda_1 + i\xi n_1 f'] + B_0 [n_2 (\lambda_2 - h) f \lambda_2 + i\xi n_2 f'] \} e^{-i \xi x} d\xi \tag{27}$$

The Fourier integral representation of $f(x)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\eta) e^{-i \eta x} d\eta$$

$$f'(x) = \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \eta \bar{f}(\eta) e^{-i \eta x} d\eta$$

therefore

$$\bar{f}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i \eta x} dx$$

$$= \frac{\pi}{r\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin r x \, e^{i \eta x} dx$$

$$= \frac{-i\pi^2}{\sqrt{2\pi}\, r} [\delta(-r - \eta) - \delta(r - \eta)].$$

Substituting the value of $f(x), f'(x)$ into (25), (26) and (27) and considering (25) only

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{ a_2 A_1 + b_2 B_1 + c_2 C_1 \} e^{-i \xi x} d\xi$$

$$= \frac{\sigma_1}{2\mu} + \frac{1}{2\pi} \iint_{-\infty}^{\infty} [\, (a_2 \lambda_1 - 2 i \eta a_3) A_0 +$$

$$(b_2 \lambda_2 - 2 i \eta b_3) B_0 + (c_2 \nu - 2 i \eta c_3) C_0 \,] \bar{f}(\eta) e^{-i(\xi + \eta) x} d\xi \, d\eta. \tag{28}$$

If we set $\xi + \eta = k$, $d\xi = dk$ then the double integral on the right hand side of (28) takes the form

$$\frac{1}{2\pi} \iint_{-\infty}^{\infty} [\, A_0 \beta_1(\xi) + B_0 \beta_2(\xi) + C_0 \beta_3(\xi) \,]_{\xi = k - \eta} \bar{f}(\eta) e^{-ikx} dk \, d\eta \tag{29}$$

where

$$\left.\begin{aligned} \beta_1(\xi) &= a_2 \lambda_1 - 2 i \eta a_3 \\ \beta_2(\xi) &= b_2 \lambda_2 - 2 i \eta b_3 \\ \beta_3(\xi) &= c_2 \nu - 2 i \eta c_3 \end{aligned}\right\} \tag{30}$$

Now replacing ξ by k on the left hand side of (28) and using (29) on the right hand side, the integral on both sides becomes single Fourier integral and taking Fourier transform on both sides of (28) and substituting the value of $\bar{j}(\eta)$ on the right hand side and using (22)

$$A_1\, a_2 + B_1\, b_2 + C_1\, c_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\, A_0\, \beta_1(\xi)$$

$$+ B_0\, \beta_2(\xi) + C_0\, \beta_3(\xi)\,]^{\xi = k-\eta} \times \frac{-i\pi^2}{\sqrt{2\pi}\, r} [\delta(-r-\eta) - \delta(r-\eta)]\, d\eta \tag{31}$$

Similarly from equations (26) and (27)

$$A_1\, a_3 + B_1\, b_3 + C_1\, c_3 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\, A_0\, \gamma_1(\xi) + B_0\, \gamma_2(\xi) +$$

$$+ C_0\, \gamma_3(\xi)]^{\xi = k-\eta} \times \frac{-i\pi^2}{\sqrt{2\pi}\, r} [\delta(-r-\eta) - \delta(r-\eta)]\, d\eta. \tag{32}$$

$$A_1\, n_1\, (\lambda_1 - h) + B_1\, n_2\, (\lambda_2 - h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [A_0\, \alpha_1(\xi)$$

$$+ B_0\, \alpha_2(\xi)]^{\xi = k-\eta} \times \frac{-i\pi^2}{\sqrt{2\pi}\, r} [\delta(-r-\eta) - \delta(r-\eta)]\, d\eta \tag{33}$$

where the left hand sides are functions of k only and

$$\left.\begin{aligned}
\gamma_1(\xi) &= a_3\, \lambda_1 + i\,\eta\, (a_1 + a_2) \\
\gamma_2(\xi) &= b_3\, \lambda_2 + i\,\eta\, (b_1 + b_2) \\
\gamma_3(\xi) &= c_3\, \nu + i\,\eta\, (c_1 + c_2) \\
\alpha_1(\xi) &= n_1\, (\lambda_1 - h)\, \lambda_1 + \xi\,\eta\, n_1 \\
\alpha_2(\xi) &= n_2\, (\lambda_2 - h)\, \lambda_2 + \xi\,\eta\, n_2
\end{aligned}\right\} \tag{34}$$

Performing the integration on the right hand side of (31), (32) and (33) sets of equations for the determinations of $A_1(k)$, $B_1(k)$ and $C_1(k)$ are obtained. Solving (31), (32) and (33)

$$A_1(k) = \frac{\pi}{2ri\, D(k)\, D(k+r)} [\, \alpha'_0 \{ n_2\, (\lambda_2 - h)\, (\gamma'_1\, c_2 - \beta'_1\, c_3) - \alpha'_1\, (b_3\, c_2 - b_2\, c_3) \}$$

$$+ \beta'_0 \{ n_2\, (\lambda_2 - h)\, (\gamma'_2\, c_2 - \beta'_2\, c_3) - \alpha'_2\, (b_3\, c_2 - b_2\, c_3) \}$$

$$+ \gamma'_0 \{ n_2\, (\lambda_2 - h)\, (\gamma'_3\, c_2 - \beta'_3\, c_3) \}]$$

$$- \frac{\pi}{2ri\, D(k)\, D(k-r)} [\, \alpha_0'' \{ n_2\, (\lambda_2 - h)\, (\gamma_1''\, c_2 - \beta_1''\, c_3)$$

$$- \alpha_1''\, (b_3\, c_2 - b_2\, c_3) \} + \beta_0'' \{ n_2\, (\lambda_2 - h)\, (\gamma_2''\, c_2 - \beta_2''\, c_3)$$

$$- \alpha_2''\, (b_3\, c_2 - b_2\, c_3) \} + \gamma_0'' \{ n_2\, (\lambda_2 - h)\, (\gamma_3''\, c_2 - \beta_3''\, c_3) \}]$$

$$B_1(k) = \frac{\pi}{2ri\, D(k)\, D(k+r)} \left[\alpha_0' \{ n_1 (\lambda_1 - h) (\gamma_1' c_2 - \beta_1' c_3) - \alpha_1' (a_3 c_2 - a_2 c_1) \}\right.$$

$$+\beta_0' \{ n_1 (\lambda_1 - h) (\gamma_2' c_2 - \beta_2' c_3) - \alpha_2' (a_3 c_2 - a_2 c_1) \}$$

$$\left. +\gamma_0' \{ n_1 (\lambda_1 - h) (\gamma_3' c_2 - \beta_3' c_3) \} \right]$$

$$- \frac{\pi}{2ri\, D(k)\, D(k-r)} \left[\alpha_0'' \{ n_1 (\lambda_1 - h) (\gamma_1'' c_2 - \beta_1'' c_3) - \alpha_1'' (a_3 c_2 - a_2 c_1)\}\right.$$

$$+\beta_0'' \{ n_1 (\lambda_1 - h) (\gamma_2'' c_2 - \beta_2'' c_3) - \alpha_2'' (a_3 c_2 - a_2 c_1) \}$$

$$\left. +\gamma_0'' \{n_1 (\lambda_1 - h) (\gamma_3'' c_2 - \beta_3'' c_2) \} \right]$$

$$C_1(k) = \frac{\pi}{2ri\, D(k)\, D(k+r)} \left[\alpha_0' \{ n_1 (\lambda_1 - h) (b_2 \gamma_1' - b_3 \beta_1')\right.$$

$$+n_2 (\lambda_2 - h) (a_3 \beta_1' - a_2 \gamma_1') + \alpha_1' (a_2 b_3 - a_3 b_2) \}$$

$$+\beta_0' \{ n_1 (\lambda_1 - h) (b_2 \gamma_2' - b_3 \beta_2') + n_2 (\lambda_2 - h) (a_3 \beta_2' - a_2 \gamma_2')$$

$$+\alpha_2' (a_2 b_3 - a_3 b_2) \} + \gamma_0' \{n_1 (\lambda_1 - h) (b_2 \gamma_3' - b_3 \beta_3')$$

$$\left. +n_2 (\lambda_2 - h) (a_3 \beta_3' - a_2 \gamma_3') \} \right]$$

$$- \frac{\pi}{2ri\, D(k)\, D(k-r)} \left[\alpha_0'' \{ n_1 (\lambda_1 - h) (b_2 \gamma_1'' - b_3 \beta_1'')\right.$$

$$+n_2 (\lambda_2 - h) (a_3 \beta_1'' - a_2 \gamma_1'') + \alpha_1'' (a_2 b_3 - a_3 b_2) \}$$

$$+\beta_0'' \{ n_1 (\lambda_1 - h) (b_2 \gamma_2'' - b_3 \beta_1'') + n_2 (\lambda_2 - h) (a_3 \beta_1'' - a_2 \gamma_1'')$$

$$\left. +\alpha_2'' (a_2 b_3 - a_3 b_2) \} + \gamma_0'' \{ n_1 (\lambda_1 - h) (b_2 \gamma_2'' - b_3 \beta_3'') + n_2 (\lambda_2 - h) a_3 \beta_3'' - a_2 \gamma_3'') \} \right]$$

where

$$\alpha_0' = \alpha_0 (k+r) \;;\; \beta_0' = \beta_0 (k+r) \;;\; \gamma_0' = \gamma_0 (k+r) \text{ etc.}$$

and

$$\alpha_0'' = \alpha_0 (k-r) \;;\; \beta_0'' = \beta_0 (k-r) \;;\; \gamma_0'' = \gamma_0 (k-r) \text{ etc.}$$

Representing $F_1(k)$, $F_2(k)$, $F_3(k)$ for the expressions within the third brackets, the value of $A_1(k)$ $B_1(k)$ and $C_1(k)$ can be written in a convenient form

$$\left.\begin{aligned} A_1(k) &= \frac{\pi}{2ri} \left[\frac{F_1(k)}{D(k)\, D(k+r)} - \frac{F_1(k)}{D(k)\, D(k-r)} \right] \\ B_1(k) &= \frac{\pi}{2ri} \left[\frac{F_2(k)}{D(k)\, D(k+r)} - \frac{F_2(k)}{D(k)\, D(k-r)} \right] \\ C_1(k) &= \frac{\pi}{2ri} \left[\frac{F_3(k)}{D(k)\, D(k+r)} - \frac{F_3(k)}{D(k)\, D(k-r)} \right] \end{aligned}\right\} \tag{35}$$

Substituting the values from (23) and (35) to (11) and (12) the component of the displacements are given by

$$u_0 = u_0^0 + \varepsilon\, u_0'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{D(\xi)} \left[-i\xi \left\{ \alpha_0(\xi)\, e^{-\lambda_1 y} + \beta_0(\xi)\, e^{-\lambda_2 y} \right\} + \nu\, \gamma_0(\xi)\, e^{-\nu y} \right] e^{-i\xi x}\, d\xi$$

$$+\frac{\varepsilon}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[-i\xi\left\{A_1(\xi)\,e^{-\lambda_1 y}+B_1(\xi)\,e^{-\lambda_2 y}\right\}+\nu\,C_1(\xi)\,e^{-\nu y}]\,e^{-i\xi x}\,d\xi \quad (36)$$

$$v_0 = v_0{}^0+\varepsilon\, v_0{}'$$

$$=\frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{D(\xi)}[\lambda_1\,\alpha_0(\xi)\,e^{-\lambda_1 y}+\lambda_2\,\beta_0(\xi)\,e^{-\lambda_2 y}+i\,\xi\,\gamma_0(\xi)\,e^{-\nu y}]\,e^{-i\xi x}\,d\xi$$

$$-\frac{\varepsilon}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\lambda_1 A_1(\xi)\,e^{-\lambda_1 y}+\lambda_2\,B_1(\xi)\,e^{-\lambda_2 y}+i\,\xi\,C_1(\xi)\,e^{-\nu y}]\,e^{-i\xi x}\,d\xi \quad (37)$$

The integral for u_0, v_0 consists of two parts, the first part *viz.*, $u_0{}^0$, $v_0{}^0$ have already been culculated for $h = 0$, $\beta^2 = 3$ and $\varepsilon' = \cdot 05$ (*c,f.* Nowacki) and we are interested for the second part for u_0, v_0 which arises due to wavy boundary. To evaluate the second parts for u_0, v_0 under $h = 0$, $\beta^2 = 3$ and $\varepsilon' = \cdot 05$, a semi-circular contour in the lower half of complex ξ plane has been considered. For $x > 0$ the integral over the semi-circular contour vanishes. Since we are interested only in surface waves the contributions from the residue of the poles have only been considered, and branch point integrals giving rise to body waves have been neglected. Therefore the part containing ε given in u_0 is

$$\frac{\pi}{\sqrt{2\pi}\;2ri}\int_{-\infty}^{\infty}\frac{F(\xi)\qquad e^{-i\xi x}}{D(\xi)\;D(\xi+r)}\,d\xi-\frac{\pi}{\sqrt{2\pi}\;2ri}\int_{-\infty}^{\infty}\frac{F(\xi)\,e^{-i\xi x}}{D(\xi)\;D(\xi-r)}\,d\xi$$

where

$$F(\xi) = -i\,\xi\,\{F_1(\xi)\,e^{-\lambda_1 y}+F_2(\xi)\,e^{-\lambda_2 y}\}+\nu\,F_3(\xi)\,e^{-\nu y}$$

and after integration

$$-\left(\frac{\pi}{2}\right)^{3/2}\frac{1}{r}\left[\frac{F(\xi_1)\,e^{-i\xi_1 x}}{D'(\xi_1)\;D(\xi_1+r)}+\frac{F(\xi_2)\,e^{-i\xi_2 x}}{D(\xi_2)\;D'(\xi_2+r)}\right.$$

$$\left.-\frac{F(\xi_1)\;\;e^{-i\xi_1 x}}{D'(\xi_1)\;D(\xi_1-r)}-\frac{F(\xi_3)\;\;e^{-i\xi_3 x}}{D(\xi_3)\;D'(\xi_3-r)}\right].$$

Again the part containing ε given in v_0:

$$\frac{\pi}{\sqrt{2\pi}\;2ri}\int_{-\infty}^{\infty}\frac{G(\xi)\quad e^{-i\xi x}}{D(\xi)\;D(\xi+r)}\,d\xi+\frac{\pi}{\sqrt{2\pi}\;2ri}\int_{-\infty}^{\infty}\frac{G(\xi)\;e^{-i\xi x}}{D(\xi)\;D(\xi-r)}\,d\xi$$

where

$$G(\xi) = \lambda_1\,F_1(\xi)\,e^{-\lambda_1 y}+\lambda_2\,F_2(\xi)\quad e^{-\lambda_2 y}+i\,\xi\,F_3(\xi)\,e^{-\nu y}$$

and after integration

$$\left(\frac{\pi}{2}\right)^{3/2}\frac{1}{r}\left[\frac{G(\xi_1)\;\;e^{-i\xi_1 x}}{D'(\xi_1)\;D(\xi_1+r)}+\frac{G(\xi_2)\,e^{-i\xi_2 x}}{D(\xi_2)\;D'(\xi_2+r)}\right.$$

$$\left.-\frac{G(\xi_1)\,e^{-i\xi_1 x}}{D'(\xi_1)\;D(\xi_1-r)}-\frac{G(\xi_3)\,e^{-i\xi_3 x}}{D(\xi_3)\;D(\xi_2)-r)}\right]$$

where $D(\xi) = 0$ gives only one real root

where $\xi_1 = \dfrac{\omega}{\cdot 9224\, c_2}$ (*cf.* Nowacki p. 429)

so $\xi_2 = \dfrac{\omega}{\cdot 9224\, c_2} - r\,;\ \xi_3 = \dfrac{\omega}{\cdot 9224\, c_2} + r.$

Therefore the final expression for u, v arising due to perturbation are

$$u_0' = -\left(\frac{\pi}{2}\right)^{3/2} \frac{1}{r}\left[\left\{\frac{1}{D(\xi_1+r)} - \frac{1}{D(\xi_1-r)}\right\}\frac{F(\xi_1)\, e^{i\omega(t-\frac{x}{c})}}{D'(\xi_1)}\right.$$

$$\left. + \frac{F(\xi_2)\, e^{i\omega\{t-x(\frac{1}{c}-\frac{r}{\omega})\}}}{D(\xi_2)\, D'(\xi_2+r)} - \frac{F(\xi_3)\, e^{i\omega\{t-x(\frac{1}{c}+\frac{r}{x})\}}}{D(\xi_3)\, D'(\xi_3-r)}\right]$$

$$v_0' = \left(\frac{\pi}{2}\right)^{3/2} \frac{1}{r}\left[\left\{\frac{1}{D(\xi_1+r)} - \frac{1}{D(\xi_1-r)}\right\}\frac{G(\xi_1)\, e^{i\omega(t-\frac{x}{c})}}{D'(\xi_1)}\right.$$

$$\left. + \frac{G(\xi_2)\, e^{i\omega\{t-x(\frac{1}{c}-\frac{r}{\omega})\}}}{D(\xi_2)\, D'(\xi_2+r)} - \frac{G(\xi_3)\, e^{i\omega\{t-x(\frac{1}{c}+\frac{r}{x})\}}}{D(\xi_3)\, D'(\xi_3-r)}\right]$$

Conclusion

From the above expressions of u_0', v_0' it is clear that if a harmonic wave moves in thermoelastic medium with a sinusoidal wavy boundary the surface waves break into three parts one part of which is the same as Rayleigh's wave in thermoelastic medium as found by Lockett and the other two parts which may be considered as scattered surface waves arising due to irregularity of the boundary, propagate with different velocities along the surface depending on the wave length as well as on the wavy nature of the boundary.

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Effect of sulphur and its compounds on the availability of manganese in soil

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Abstract

The elemental form of sulphur is more efficient than other compounds of sulphur in making manganese available in soil.

The effect of sulphur on the metabolism of manganese in both plants as well as soil has been shown by a number of workers. Ames and Boltz (1919) reported that addition of sulphur hindered the nitrification process in the soil due to the development of acidity and increase of its Mn content. Bertramson, Fried and Tisdale (1950) stated that sulphur may affect the availability of different nutrients in several ways, such as by lowering the pH, by ionic relationship when it forms $SO_4^{=}$ ions, and by serving as reductant or electron donor. Conard (1950) studied the effect of elemental sulphur and that added as sulphates on the availability of soil manganese but his data were inconclusive as to whether the release of nutrient by gypsum was caused by sulphur or calcium.

Greer, Pringle, and Kant (1952) showed that availability of manganese by plants was directly proportional to the pH of the soil. Tisdale and Bertramson (1950) showed that the increased sulphate-sulphur content in the solution causes increased absorption of manganese by plants.

The present investigation was undertaken with a view to studying the effect of elemental sulphur and its compounds on the availability of manganese in the soil.

Method and Material

Three soils namely Matkota clay loam, pH 8·2, Kanpur No. 2, pH 9·3, and one sample from Government Agricultural College Farm, pH 8·1 were taken for study. Samples were prepared and different doses of sulphur either elemental or as its compounds were added and mixed thoroughly. The moisture content of the soil was regulated according to the optimum field conditions from time to time. Thorough stirring of the soil was done in order to break the lumps and to provide sufficient aeration. The sampling of the soil was done after an interval of 10, 20 and 40 days respectively.

Manganese in exchangeable form was determined colorimetrically by extracting the soil with ammonium-acetate as described by Johnson Ulrich (1959).

Result and Discussions

Data in table Nos. 1, 2 and 3 show that the exchangeable Mn contents in all the three soils were higher in the case of elemental sulphur treatments in comparison to the others. The Mn content increased with the increase in the doses of sulphur added. It may be attributed to the reduction of the pH of soil and ultimately increases the availability of divalent Mn ions. Similar observations were made by Shedd (1914), Bertramson, Fried and Tisdale (1950) and Morris (1948).

The treatment with ammonium sulphate was next in liberating exchangable Mn. The efficiency of gypsum and superphosphate containing $CaSO_4$ was almost of the same order. In the case of gypsum treatment the increase was not in accordance with the doses of sulphur added through gypsum. Similar observations were made by Teucher and Adler (1960), and Morris and Pierre (1947). No significant increase was observed in the availability of Mn in the case of K_2SO_4 treatment.

TABLE 1

Matkota clay loam

Treatment	Doses of sulphur in ppm	10 days		20 days		40 days	
		Mn ppm	pH	Mn ppm	pH	Mn ppm	pH
Blank	–	80	8·2	80	8·2	80	8·2
Control	–	82	8·2	83	8·2	84	8·2
Elemental 'S'	50	90	8·2	100	8·2	106	8·1
Elemental 'S'	400	91	7·9	102	7·8	108	8·1
Elemental 'S'	800	92	7·7	102	7·6	119	8·1
Elemental 'S'	1600	98	7·7	103	7·7	128	8·0
Gypsum	50	84	8·2	92	8·2	105	8·2
Gypsum	400	86	8·1	95	8·1	106	8·2
Gypsum	800	86	8·0	96	8·0	108	8·2
Gypsum	1600	89	7·8	98	7·7	112	7·9
Ammonium sulphate	400	88	8·0	96	8·0	108	8·2
Superphosphate containing $CaSO_4$	400	84	7·8	93	7·9	105	8·2
Potassium sulphate	400	82	8·1	90	8·2	103	8·2

TABLE 2

Kanpur No. 2

Treatments	Doses of sulphur in ppm	10 days		20 days		40 days	
		Mn ppm	pH	Mn ppm	pH	Mn ppm	pH
Blank	–	6	9·3	7	9·3	10	9·3
Control	–	7	9·3	8	9·3	12	9·3
Elemental 'S'	50	8	9·2	10	9·2	13	9·3
Elemental 'S'	400	10	9·0	13	9·0	16	9·1
Elemental 'S'	800	12	8·9	15	8·9	20	9·0
Elemental 'S'	1600	16	8·8	20	8·7	30	8·8
Gypsum	50	10	9·2	10	9·2	12	9·3
Gypsum	400	11	9·0	12	9·1	13	9·1
Gypsum	800	12	8·7	12	8·8	14	9·0
Gypsum	1600	12	8·6	12	8·6	14	8·7
Ammonium sulphate	400	9	8·8	9	9·0	12	9·0
Superphosphate containing $CaSO_4$	400	9	9·0	12	9·1	19	9·2
Potassium sulphate	400	8	9·3	13	9·3	15	9·3

TABLE 3

Agricultural College Farm Soil

Treatments	Doses of sulphur in ppm	10 days		20 days		40 days	
		Mn ppm	pH	Mn ppm	pH	Mn ppm	pH
Blank	–	10	8·1	12	8·1	14	8·2
Control	–	11	8·1	16	8·2	20	8·2
Elemental 'S'	50	15	8·0	35	8·2	46	8·2
Elemental 'S'	400	19	7·8	40	7·9	51	8·1
Elemental 'S'	800	18	7·6	48	7·8	58	7·9
Elemental 'S'	1600	25	7·5	55	7·6	69	7·7
Gypsum	50	14	8·0	18	8·1	23	8·2
Gypsum	400	12	7·8	22	7·9	33	7·9
Gypsum	800	15	7·6	29	7·7	39	7·7
Gypsum	1600	12	7·6	31	7·7	48	7·6
Ammonium sulphate	400	18	7·7	30	7·8	39	7·9
Superphosphate containing $CaSO_4$	400	10	7·6	26	8·0	35	8·1
Potassium sulphate	400	9	7·8	20	8·0	28	8·2

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A New Alkaloid from the Seeds of *Erythrina lithosperma*

By

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Abstract

A new alkaloid. $C_{30}H_{36}O_5N_4$, provisionally named as 'Erythrinine', has been isolated from the seeds of *Erythrina lithosperma*. It has been found to contain four $-OCH_3$ groups and one -CO-NH group. All the nitrogen atoms have been found to be present in ring systems.

Erythrina lithosperma[1] (N.O. Leguminosae) is a large tree found abundantly in Burma up to a height of 3000 ft. The genus, Erythrina, consists of many important medicinal plants *e.g. E. indica*, *E. suberosa* etc. The bark and the root of these plants are of immense use in the treatment of opthalmia and rheumatism.

A good number of alkaloids[2,3,4,5,6], have been isolated from various species of this genus. On reviewing the literature, it becomes quite obvious that inspite of good amount of investigations directed to the study of erythrina alkaloids, not much headway could be possible in regard to the elucidation of the structures of these alkaloids. Moreover, no work seems to have been done on the chemical study of the plant *Erythrina lithosperma*. In view of the need and importance of such a study, the authors took up the chemical examination of its seeds. The present paper deals with the isolation of the alkaloid and the study of the nature of the hetero-atoms present in it. The results of the detailed studies in regard to the elucidation of structure will be communicated later.

Experimental

Isolation of the Alkaloid: The defatted seeds were extracted with ethanol under reflux. The solvent was distilled off and the residue thus obtained redissolved in methanol. Solvent ether was added in excess ; the brown precipitate then obtained was removed by filtration and studied separately.

The yellowish filtrate was concentrated to half of its volume and kept in a refrigerator overnight when a crystalline compound was obtained. On purification by T. L. C., the compound melted at 253–4°).

Properties and Characteristics of the base: It was found to be soluble in methanol, ethanol, water and dil. mineral acids. It gave positive tests with alkaloidal reagents[7] showing it to be an alkaloid.

Found, C = 68·24 ; H = 6·71 ; N = 10·63% (Dumas); Mol. wt. = 532 (mass spectrum) ; Calculated for $C_{30}H_{36}O_5N_4$, C = 67·64 ; H = 6·76 ; N 10·53%, Mol. wt. = 532. Thus, the molecular formula of the alkaloid is $C_{30}H_{36}O_5N_4$.

The alkaloid gave a number of derivatives such as methiodide, m.p. 201–2°, picrate, m.p. 105°, hydrochloride, m.p. 120°, gold chloride (m.p. above 360°) and platinic chloride (m.p. above 360°).

Nature and Environment of Nitrogen atoms : As no nitrogen was detected by Kjeldahl's method, all the nitrogen atoms are present in the ring systems.

Herzig-Meyer hydriodic acid treatment of the base did not give any alkyliodide indicating the absence of N-alkyl group in the base.

The fromation of ammonia on fusion of the base with solid KOH, indicates the presence of $-CO-NH-$ (or $-CONH_2$) group. The strong absorption peak at 1635 cm^{-1} in the I. R. further confirms the presence of such a grouping.

Nature of Oxygen atoms : The base did not give positive tests for the presence of

$-OH$, $-CHO$, $-COOH$ and $CH_2\begin{matrix} /O- \\ \backslash O- \end{matrix}$ groups.

The Viebock and Brecher method[8] of alkoxy determination showed the presence of four $-OCH_3$ groups in the base. The absorption peaks at 1355 cm^{-1} and 1380 cm^{-1} (in the I. R.) clearly indicate the presence of methoxyl groups. Thus, four oxygen atoms have been accounted for. The fifth oxygen atom is more probably present at -CO–NH.

Oxidative and degradative studies are in progress and the results will be communicated in subsequent publication.

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A Generalised integral transform of two complex variables

By

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Abstract

This generalisation of Laplace transform in two variables is mainly based on the transform given by Sharma[2]. In later section, an inversion formula is established and the theorem which gives the solution of the integral equation, is proved by applying Mellin-inversion formula. The change in the order of integration is justified by de la Vallee Poussin's theorem.

1. Introduction. Recently, Verma[3] defined a new transform which generalises various known generalisations of Laplace transform given in previous years.

In the present note I obtain a generalisation on the basis of the transform[3]. Also, I give an inversion formula analogous to Sharma[2]. The beauty of this transform lies in the fact that it includes most of the known transforms as particular cases. It is expected that the further study of the transform should yield general and more interesting results.

2. Definition. A generalised integral transform $\phi^{m,\, n}_{m^1,\, n^1}[\lambda, \mu]$ is defined by the relation

$$\phi\left[f(x,y)\ \lambda;\ \mu:\ \begin{matrix} m\,;\, n \\ m^1;\, n^1 \end{matrix} : \begin{matrix} (a_p);\ (b_q) \\ (\alpha_p{}^1);\ (\beta_q{}^1) \end{matrix}\right] = \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\lambda x - \frac{1}{2}\mu y} \tag{2·1}$$

$$\times\, G^{m,\, n}_{p,\, q}\left(\lambda^2 x^2 \,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right) G^{m^1,\, n^1}_{p^1,\, q^1}\left(\mu^2 y^2 \,\middle|\, \begin{matrix} (\alpha_p{}^1) \\ (\beta_q{}^1) \end{matrix}\right) f(x, y)\, dx\, dy,$$

where $G^{m,\, n}_{p,\, q}\left(\lambda^2 x^2 \,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right)$ and $G^{m^1,\, n^1}_{p^1,\, q^1}\left(\mu^2 y^2 \,\middle|\, \begin{matrix} (\alpha_p{}^1) \\ (\beta_q{}^1) \end{matrix}\right)$ are Meijer's G-functions.

As usual (ε_p) denotes the sequence of elements

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p.$$

The above defined transform exists when $m, n, p, q, m^1, n^1, p^1$ and q^1 are non-negative integers, such that $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, $0 \leqslant m^1 \leqslant q^1$, $0 \leqslant n^1 \leqslant p^1$,

$p+q < 2(m+n)$, $\quad |\arg \lambda| < \pi\,[m+n-\tfrac{1}{2}(p+q)]$, $p^1+q^1 < 2(m^1+n)$,

$|\arg \mu| < \pi\,[m^1+n^1-\tfrac{1}{2}(p^1+q^1)]$

and $\quad x^{2(b_m)}\, y^{2(\beta_{m^1})}\, f(x,y) \in L(0,\infty)$.

3. The inversion formula. The following is the theorem which gives the solution of the integral equation (2·1), solved for $f(x, y)$.

Theorem. If

$$\phi\left[f(x,y) : \lambda\,;\,\mu : \begin{matrix} m\,;\,n \\ m^1;n^1 \end{matrix} : \begin{matrix} (a_p)\,;\,(b_q) \\ (\alpha_p{}^1)\,;\,(\beta_q{}^1) \end{matrix}\right] = \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\lambda x - \frac{1}{2}\mu y} \times$$

$$\times\, G^{m,\,n}_{p,\,q}\left(\lambda^2 x^2 \,\middle|\, \begin{matrix}(a_p)\\(b_q)\end{matrix}\right) G^{m^1,\,n^1}_{p^1,\,q^1}\left(\mu^2 y^2 \,\middle|\, \begin{matrix}(\alpha_p{}^1)\\(\beta_q{}^1)\end{matrix}\right) f(x,y)\,dx\,dy,$$

and

(3·1) $$h(x,y) = \frac{1}{(2\pi i)^2}\int_{c-i\infty}^{c+i\infty}\int_{c^1-i\infty}^{c^1+i\infty} \frac{x^{-r}\,y^{-s}}{G_1(1-r)\,G_2(1-s)}\,dr\,ds,$$

where

(3·2) $$G_1(r) = \pi^{-\frac{1}{2}}\, 2^{2r-1}\, G^{m,\,n+2}_{p+2,\,q}\left(16 \,\middle|\, \begin{matrix}\frac{1}{2}-\frac{1}{2}r,\ 1-\frac{1}{2}r,\ (a_p)\\ (b_q)\end{matrix}\right)$$

and

(3·3) $$G_2(s) = \pi^{-\frac{1}{2}}\, 2^{2s-1}\, G^{m^1,\,n^1+2}_{p^1+2,\,q^1}\left(16 \,\middle|\, \begin{matrix}\frac{1}{2}-\frac{1}{2}s,\ 1-\frac{1}{2}s,\ (\alpha_p{}^1)\\ (\beta_q{}^1)\end{matrix}\right),$$

then

(3·4) $$f(x,y) = \int_0^\infty\int_0^\infty h(x\lambda,\mu y)\ \phi\left[f(x,y) : \lambda;\,\mu : \begin{matrix} m\,;\,n \\ m^1;\,n^1 \end{matrix} : \begin{matrix} (a_p),\,(b_q) \\ (\alpha_p{}^1),\,(\beta_q{}^1) \end{matrix}\right] d\lambda\,d\mu,$$

provided that $|h(x,y)|$ exists, (3·4) is convergent,

$$\lambda^{-c}\,\mu^{-c^1}\,\phi\left[f(x,y) : \lambda;\mu : \begin{matrix} m\,;\,n \\ m^1;\,n^1 \end{matrix} : \begin{matrix} (a_p);\,(b_q) \\ (\alpha_p{}^1);\,(\beta_q{}^1) \end{matrix}\right] \in L(0,\infty),\ x^{c-1}\,y^{c^1-1} f(x,y) \in L(0,\infty),$$

$c < R\,(2\,b_i)+1$, $(i = 1, 2, \ldots, m)$ and $c^1 < R\,(2\beta_j)+1$, $(j = 1, 2, \ldots, m^1)$.

Proof. Substituting the value from (2·1), it is seen

$$\int_0^\infty\int_0^\infty \lambda^{-1}\,\mu^{-s}\ \phi\left[f(x,y) : \lambda\,;\,\mu : \begin{matrix} m;\,n \\ m^1;\,n^1 \end{matrix} : \begin{matrix} (a_p)\,;\,(b_q) \\ (\alpha_p{}^1)\,;\,(\beta_q{}^1) \end{matrix}\right] d\lambda\,d\mu$$

$$= \int_0^\infty\int_0^\infty \lambda^{-r}\,\mu^{-s}\left\{\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\lambda x-\frac{1}{2}\mu y}\, G^{m,\,n}_{p,\,q}\left(\lambda^2 x^2\,\middle|\,\begin{matrix}(a_p)\\ b_q)\end{matrix}\right) G^{m^1,\,n^1}_{p^1,\,q^1}\left(\mu^2 y^2\,\middle|\,\begin{matrix}(\alpha_p{}^1)\\(\beta_q{}^1)\end{matrix}\right)\right.$$

$$\left.\times\, f(x,y)\,dx\,dy\right\}d\lambda\,d\mu,$$

which on change of order of integration and simple substitution yields

$$\int_0^\infty \int_0^\infty \lambda^{-r}\mu^{-s}\, \phi\left[f(x,y) : \lambda ; \mu : \begin{matrix} m; n \\ m^1, n^1 \end{matrix} : \begin{matrix} (a_p); (b_q) \\ (\alpha_{p^1}); (\beta_{q^1}) \end{matrix}\right] d\lambda\, d\mu = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} f(xy)\, dxdy$$

$$\times \int_0^\infty \int_0^\infty e^{-\frac{1}{2} u - \frac{1}{2} u^1}\, u^{-r}\, u^{1-s}\, G^{m,n}_{p,q}\left(u^2 \left| \begin{matrix} a_p) \\ b_q) \end{matrix}\right.\right) G^{m^1, n^1}_{p^1, q^1}\left(u^{1^2} \left| \begin{matrix} (\alpha_{p^1}) \\ (\beta_{q^1}) \end{matrix}\right.\right) du\, du^1.$$

Since x, y – integrals and u, u^1 – integrals are independent of each other, on evaluating the inner integrals and to which on applying Mellin-inversion formula, (3·4) is obtained.

The change of order of integration is justified by de la Vallee Poussin's theorem (1)

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A theorem on Varma Transform* †

By

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Abstract

A theorem connecting the well-known Varma and Meijer Transforms has been proved by utilizing the integral representation of the Whittaker function $W_{k,m}(z)$. The theorem and its corollary have been used in evaluating certain infinite integrals involving ${}_2F_1$, F_4 and F_c.

1. Introduction

In 1951, Varma[11] gave the second generalization of the well-known Laplace transform

(1·1) $$\phi(p) = p\int_0^\infty e^{-pt} f(t)\, dt,\ R(p) > 0,$$

in the form

(1·2) $$\phi(p) = p\int_0^\infty (pt)^{m-1/2} e^{-pt/2} W_{k,m}(pt) f(t)\, dt,\ R(p) > 0.$$

When $k+m = \frac{1}{2}$, (1·2) reduces to (1·1) by virtue of the identity.

$$W_{\mu+1/2,\, \pm\mu}(x) = x^{\mu+1/2} e^{-x/2}.$$

Meijer[5], in the year 1940, gave his first generalization of (1·1) in the following form :

(1·3) $$\phi(p) = \left(\frac{2}{\pi}\right)^{1/2} p\int_0^\infty (pt)^{1/2} K_\nu(pt) f(t)\, dt,\ R(p) > 0.$$

When $\nu = \pm\frac{1}{2}$, (1·3) reduces to (1·1) due to

$$K_{\pm 1/2}(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}.$$

We shall denote (1·1), (1·2) and (1·3) symbolically as

$$\phi(p) \doteq f(t),\ \ \phi(p) \ \frac{\overline{V}}{k,m}\ f(t) \ \text{ and } \ \phi(p) \ \underset{\nu}{\overset{k}{\doteq}}\ f(t) \ \text{ respectively.}$$

The aim of this paper is to prove a theorem connecting the Varma's transform (1·2) and the Meijer's Bessel transform (1·3) by utilizing the integral representation of the Whittaker function $W_{k,m}(z)$. The theorem and its corollary

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have been utilized in evaluating certain infinite integrals involving Gauss's hypergeometric function ${}_2F_1$, Appell's function F_4 and Lauricella's function F_C.

2. *Theorem :*

If
$$\phi(p) \underset{\nu}{\overset{k}{=}} f(t),$$
then

$$(2{\cdot}1) \qquad p^{\nu+1/2} \int_0^\infty x^{-k} (x+b)^{k-1} P^{2k}_{\nu-1/2}\left(\sqrt{1+\tfrac{x}{b}}\right) \phi\left(p\sqrt{1+\tfrac{x}{b}}\right) dx \underset{2k,\nu}{\overset{V}{=}} 2\, t^{-\nu-1/2} f(t),$$

provided that $R(k-\frac{1}{2}) < 0$, $R(b) > 0$, $R(p) > 0$, $R(\frac{1}{2} \pm \nu+\xi) > 0$,

where $f(t) = 0(t^{\xi})$ for small x.

Proof: We have
$$\phi\{p\sqrt{1+x/b}\} = \left(\frac{2}{\pi}\right)^{1/2} \{p\sqrt{1+x/b}\}^{3/2} \int_0^\infty t^{1/2} K_\nu\{pt\sqrt{1+x/b}\} f(t)\, dt.$$

Multiplying both sides by
$$x^{-k}(x+b)^{k-1} P^{2k}_{\nu-1/2}\left(\sqrt{1+x/b}\right),$$
integrating for x from 0 to ∞ and interchanging the order of integration, we have
$$\int_0^\infty x^{-k}(x+b)^{k-1} P^{2k}_{\nu-1/2}\left(\sqrt{1+x/b}\right) \phi\{p\sqrt{1+x/b}\}\, dx$$
$$= \left(\frac{2}{x}\right)^{\frac{1}{2}} p^{3/2} b^{-3/4} \int_0^\infty t^{\frac{1}{2}} f(t) \left[\int_0^\infty x^{-k}(x+b)^{k-1/4} P^{2k}_{\nu-1/2}\left(\sqrt{1+x/b}\right) K_\nu\{pt\sqrt{1+x/b}\}\, dx\right] dt.$$

Evaluating the x–integral by [8, p. 158]
$$\int_0^\infty t^{-k}(t+b)^{k-1/4} P^{2k}_{2m-1/2}\left(\sqrt{1+t/b}\right) K_{2m}\{2\sqrt{p(t+b)}\}\, dt$$
$$= 2^{-1} p^{-3/4} \pi^{1/2} e^{-\sqrt{bp}} W_{2k,\,2m}(2\sqrt{bp}),$$

for $\qquad R(k-\frac{1}{2}) < 0, \quad R(p) > 0, \quad R(b) > 0,$

we arrive at (2·1).

Regarding the change of the order of integration, we must have

(*i*) the x–integral absolutely convergent. This is so if $R(k-\frac{1}{2}) < 0$, $R(p) > 0$, $R(b) > 0$.

(*ii*) the t–integral absolutely convergent. This is so if $R(p) > 0$, $R(3/2 \pm \nu+\xi) > 0$, where $f(t) = 0\,(t^{\xi})$ for small t; and

(*iii*) any one of the resulting integrals absolutely convergent :

for R. H. S. to exist, $R(p) > 0$, $R(\frac{1}{2} \pm \nu + \xi) > 0$.

Thus the change of the order of integration is admissible for the conditions stated with the theorem by de la Vallée Poussin theorem.

Example :

Taking

$$f(t) = t^{2k} \exp\{-(a^2+c^2)\, t\}\, I_\sigma\,(2\, act),$$

we find that [2, p. 18]

$$\phi(p) = 2^{2k}\, \pi^{-1} \sum_{\lambda=\frac{1}{2},\, -\frac{1}{2}} \frac{(2ac)^\sigma\, (a^2+c^2)^{\lambda+\frac{1}{2}}\, \Gamma(-\lambda)\, \Gamma\{\frac{1}{2}\,(2k+2+\lambda\pm\nu+\sigma)\}}{p^{2k+\sigma+\lambda+\frac{1}{2}}\, \Gamma(1+\sigma)}$$

$$F_4\left[\tfrac{1}{2}(2k+2+\lambda+\sigma-\nu),\, \tfrac{1}{2}(2k+2+\lambda+\sigma+\nu)\,;\, 1+\sigma,\, 1+\lambda\,; \frac{4a^2c^2}{p^2}\,,\, \frac{(a^2+c^2)^2}{p^2}\right],$$

for $\quad R(2k+\sigma\pm\nu+3/2) > 0, \quad R[p+(a\pm c)^2] > 0\,;$

and [9, p. 174]

$$2p^{\nu+\frac{1}{2}}\int_0^\infty t^{2k-1} \exp\left[-\left(\frac{p}{2}+a^2+c^2\right) t\right] W_{2k,\nu}(pt)\, I_\sigma(2\, act)\, dt$$

$$= 2(ac)^\sigma\, \Gamma(\tfrac{1}{2}+2k\pm\nu+\sigma)\, p^{-2k-\sigma+\nu+\frac{1}{2}}\, [\Gamma(\sigma+1)]^{-2}$$

$$F_4\left[\tfrac{1}{2}+2k-\nu+\sigma,\, \tfrac{1}{2}+2k+\nu+\sigma\,;\, \sigma+1,\, \sigma+1; -\frac{a^2}{p}, -\frac{c^2}{p}\right],$$

for $\quad R(\frac{1}{2}\pm\nu+\sigma+2k) > 0, \quad R[\,p+(a\pm c)^2\,] > 0.$

Hence (2·1) yields

$$(2{\cdot}2) \qquad \sum_{\lambda=\frac{1}{2},\, -\frac{1}{2}} \frac{2^{\,2k+\sigma-1}\, (a^2+c^2)^{\lambda+\frac{1}{2}}\, \Gamma(-\lambda)\, \Gamma(\sigma+1)\, \Gamma\{k+1+\frac{1}{2}\,(\lambda+\sigma\pm\nu)\}}{\pi\, p^{\lambda+\frac{1}{2}}\, b^{-k-\sigma/2-\lambda/2-1/4}\, \Gamma(\frac{1}{2}+2k\pm\nu+\sigma)}$$

$$\int_0^\infty x^{-k}\, (x+b)^{-\sigma/2-\lambda/2-5/4}\; P^{\;2k}_{\nu-1/2}\left(\sqrt{1+x/b}\right)$$

$$F_4\left[k+1+\frac{\sigma+\lambda-\nu}{2},\, k+1+\frac{\sigma+\lambda+\nu}{2}\,;\, 1+\sigma,\, 1+\lambda\,;\, \frac{4\, a^2c^2}{p^2(1+x/b)}\,,\, \frac{(a^2+c^2)^2}{p^2(1+x/b)}\right] dx$$

$$= F_4\left[\tfrac{1}{2}+2k+\sigma-\nu,\, \tfrac{1}{2}+2k+\sigma+\nu;\, \sigma+1,\, \sigma+1\,;\, -\frac{a^2}{p}, -\frac{c^2}{p}\right],$$

for $\; R(k-\frac{1}{2})<0,\; R(b)>0,\; R(p)>0,\; R(\frac{1}{2}\pm\nu+2k+\sigma)>0,\; R(\sqrt{p}\,)>|\,I_m(a)\,|+|\,I_m(c)\,|$

As $c \to 0$, (2·2) gives the following, on using [7, p. 72] and [3, p. 126 (20)].

$$(2{\cdot}3) \qquad \int_0^\infty x^{-2k}\, (x+b)^{-\sigma/2-1}\; {}_2F_1\!\left(\frac{1}{4}+\frac{\nu}{2}-k,\, \frac{1}{4}-\frac{\nu}{2}-k;\, 1\text{-}2k; -\frac{x}{b}\right)$$

$${}_2F_1\left[k+\frac{\sigma}{2}+\frac{3}{4}-\frac{\nu}{2},\, k+\frac{\sigma}{2}+\frac{3}{4}+\frac{\nu}{2} : 2k+\sigma+2;\, 1-\frac{a^4b}{p^2(x+b)}\right] dx$$

$$= \frac{\pi\,\Gamma(1-2k)\,\Gamma(2k+\sigma+2)\,\Gamma(\frac{1}{2}+2k\pm\nu+\sigma)}{b^{2k+\sigma/2}\,2^{4k+\sigma-1}\,\Gamma(1+\sigma)\,\Gamma(k+\sigma/2\pm\nu/2+1\pm 1/4)}$$

$${}_2F_1\left[\tfrac{1}{2}+2k-\nu+\sigma,\ \tfrac{1}{2}+2k+\nu+\sigma\ ;\ \sigma+1\ ;\ -\frac{a^2}{p}\right],$$

for $\quad R(k-\frac{1}{2}) < 0\,,\quad R(b) > 0,\quad R(p) > 0,\quad R(\frac{1}{2}\pm\nu+2k+\sigma) > 0$

As $a \to 0$, (2·3) reduces to a known result [4, p. 400 (9)].

Taking $2k+\nu = \frac{1}{2}$, and using

$$P_\nu^{-\nu}(z) = 2^{-\nu}\,(z^2-1)^{\nu/2}\,[\Gamma(\nu+1)]^{-1},$$

(2·1) Yields the following :

Corollary

If $$\phi(p) \overset{k}{\underset{\nu}{\doteq}} f(t),$$

then

$$(2\cdot4)\qquad p^{\nu+1/2}\int_0^\infty x^{\nu-1/2}\,(x+b)^{-\nu/2-3/4}\ \phi\left\{p\ \sqrt{1+x/b}\right\}dx$$

$$\doteq 2^{\nu+1/2}\,b^{\nu/2-1/4}\,\Gamma(\nu+\tfrac{1}{2})\,t^{-\nu-1/2}f(t),$$

provided that $R(p) > 0$, $R(b) > 0$, $R(\nu+\frac{1}{2}) > 0$ and $R(\frac{1}{2}\pm\nu+\xi) > 0$ where $f(t) = 0\,(\,t^{\xi}\,)$ for small x.

Example :

Let

$$f(t) = t^{\sigma-3/2}\prod_{i=1}^{n}\,[I_{\nu i}\,(\alpha_i\,t)],$$

then [10]

$$\phi(p) = 2^{\sigma-1/2}\,\pi^{-1/2}\,p^{-\Sigma\,\nu_i-\sigma+3/2}\ \Gamma\{\tfrac{1}{2}\,(\sigma+\Sigma\,\nu_i\pm\nu)\}\prod_{i=1}^{n}(a_i^{\nu_i})\prod_{i=1}^{n}\,[\Gamma(1+\nu_i)]^{-1}$$

$$F_c\left[\tfrac{1}{2}(\sigma+\Sigma\nu_i-\nu),\ \tfrac{1}{2}(\sigma+\Sigma\nu_i+\nu)\ ;\ 1+\nu_1,\ldots,1+\nu_n\ ;\ \frac{a_1^2}{p^2},\ldots,\frac{\alpha_n^2}{p^2}\right],$$

where F_c is one of the four Lauricella's hypergeometric functions of n variables [1, p. 114] and $R(\sigma\pm\nu+\Sigma\nu_i) > 0$, $R(p) > \sum_{i=1}^{n}|\,R(a_i)\,|$.

Here $\Sigma\nu_i$ stands for $\nu_1+\nu_2+\ldots+\nu_n$.

Hence (2·4) gives

(2·5) $\int_0^\infty x^{\nu-1/2}\left(1+\frac{x}{b}\right)^{-\frac{1}{2}(\Sigma\,\nu_i+\sigma+\nu)} F_C\Big[\tfrac{1}{2}(\sigma+\Sigma\,\nu_i-\nu),\ \tfrac{1}{2}(\sigma+\Sigma\,\nu_i+\nu)\,;$

$$1+\nu_1,\ldots,1+\nu_n;\frac{\alpha_1^2}{p^2(1+x/b)},\ldots,\frac{\alpha_n^2}{p^2(1+x/b)}\Big]\,dx$$

$$= b^{\nu+1/2}\,\Gamma(\nu+\tfrac{1}{2})\,\Gamma\{\tfrac{1}{2}(\sigma-\nu-1+\Sigma\,\nu_i)\}\,[\Gamma\{\tfrac{1}{2}(\sigma+\nu+\Sigma\,\nu_i)\}]^{-1}$$

$$F_C\Big[\tfrac{1}{2}(\sigma-\nu+\Sigma\,\nu_i),\tfrac{1}{2}(\sigma-\nu+\Sigma\,\nu_i-1):1+\nu_1,\ldots,1+\nu_n;\frac{\alpha_1^2}{p^2},\ldots,\frac{\alpha_n^2}{p^2}\Big],$$

for $R(b)>0,\quad R(\nu+\tfrac{1}{2})>0,\quad R(\sigma+\nu+\Sigma\,\nu_i)>0,\quad R(\sigma-\nu-1+\Sigma\,\nu_i)>0,$

$$R(p)>\sum_{i=1}^{n}|R(\alpha_i)|.$$

When $n=2$, F_C breaks up into F_4 and we have

(2·6) $$\int_0^\infty x^{\nu-1/2}\left(1+\frac{x}{b}\right)^{-\frac{1}{2}(\nu_1+\nu_2+\sigma+\nu)}$$

$$F_4\Big[\tfrac{1}{2}(\sigma+\nu_1+\nu_2-\nu),\tfrac{1}{2}(\sigma+\nu_1+\nu_2+\nu);1+\nu_1,1+\nu_2;\frac{\alpha_1^2}{p^2(1+x/b)},\frac{\alpha_2^2}{p^2(1+x/b)}\Big]dx$$

$$= b^{\nu+1/2}\,\Gamma(\nu+\tfrac{1}{2})\,\Gamma\{\tfrac{1}{2}(\sigma-\nu-1+\nu_1+\nu_2)\}\,[\Gamma\{\tfrac{1}{2}(\sigma+\nu+\nu_1+\nu_2)\}]^{-1}$$

$$F_4\Big[\tfrac{1}{2}(\sigma-\nu+\nu_1+\nu_2),\tfrac{1}{2}(\sigma-\nu-1+\nu_1+\nu_2);1+\nu_1,1+\nu_2;\frac{\alpha_1^2}{p^2},\frac{\alpha_2^2}{p^2}\Big],$$

for $R(\sigma+\nu+\nu_1+\nu_2)>0,\ R(\sigma-\nu-1+\nu_1+\nu_2)>0,\ R(p)>|R(\alpha_1)|+|R(\alpha_2)|.$

As $\alpha_2\to 0$, (2·6) yields a particular case of a known result [6, p. 386].

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